



The Characteristic Function for Complex Doubly Infinite Jacobi Matrices

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Abstract. We introduce a class of doubly infinite complex Jacobi matrices determined by a simple convergence condition imposed on the diagonal and off-diagonal sequences. For each Jacobi matrix belonging to this class, an analytic function, called a characteristic function, is associated with it. It is shown that the point spectrum of the corresponding Jacobi operator restricted to a suitable domain coincides with the zero set of the characteristic function. Also, coincidence regarding the order of a zero of the characteristic function and the algebraic multiplicity of the corresponding eigenvalue is proved. Further, formulas for the entries of eigenvectors, generalized eigenvectors, a summation identity for eigenvectors, and matrix elements of the resolvent operator are provided. The presented method is illustrated by several concrete examples.

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1. Introduction

In a recent paper [15], a method for the spectral analysis of a certain class of semi-infinite Jacobi matrices based on the so-called characteristic function was developed. The spectral analysis of semi-infinite Jacobi matrices is intimately related to classical branches of analysis such as orthogonal polynomials, moment problems, and continued fractions. There are many monographs that have focused on this, see [2, 12, 17] for several examples. On the other hand, the spectral theory of doubly infinite Jacobi matrices does not appear as often; a nice exposition of this theory can be found in [4, Chap. 7]. Some aspects of the spectral theory of the doubly and semi-infinite Jacobi matrices are quite similar (one may consult, for example, [10]); however, some are different. The main difference is the fact that, in the case of doubly infinite matrices, the space of the solutions to the corresponding second-order difference equation is two-dimensional.

The main aim of this article is twofold. First, in analogy with the semi-infinite case treated in [15], we introduce the characteristic function associated with the doubly infinite Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & & \\ & w_{-2} & \lambda_{-2} & w_{-1} & & & & \\ & & w_{-1} & \lambda_{-1} & w_0 & & & \\ & & & w_0 & \lambda_0 & w_1 & & \\ & & & & w_1 & \lambda_1 & w_2 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (1)$$

where $\lambda_n, w_n \in \mathbb{C}$ and $w_n \neq 0$ for all $n \in \mathbb{Z}$. We also show how the characteristic function can be used to analyze spectral properties of a linear operator acting on $\ell^2(\mathbb{Z})$ whose matrix representation with respect to the standard basis of $\ell^2(\mathbb{Z})$ coincides with \mathcal{J} . Second, we extend the method by proving a result concerning the algebraic multiplicity of eigenvalues and generalized eigenvectors.

The subclass of matrices \mathcal{J} for which the characteristic function is well defined is determined by a simple convergence condition imposed on λ_n and w_n , see (8). The self-adjointness of an operator associated with \mathcal{J} is not essential for the presented method; therefore, we treat the matrix \mathcal{J} with complex entries, which might be of interest from the point of view of the spectral theory of non-self-adjoint operators - a currently very active and rapidly developing field [6, 8, 18]. The main results of this paper are stated in Theorems 14 and 20.

In more detail, we show [under a mild assumption additional to the essential convergence condition (8)] that the matrix \mathcal{J} uniquely determines a densely defined closed (Jacobi) operator J whose spectral properties are related to the properties of the corresponding characteristic function. Namely, by being restricted to a certain subset of \mathbb{C} , the spectrum of J in this set is discrete and coincides with the set of zeros of the characteristic function. Further, we provide formulas for eigenvectors, a summation formula for the entries of an eigenvector and an expression for the entries of the matrix representation of the resolvent operator. These results are worked out within Sect. 2 and represent a doubly infinite analog to the corresponding results derived in [15] for the case of semi-infinite Jacobi matrices.

Section 3 is devoted to the connection between the order of zeros of the characteristic function and the algebraic multiplicity of the corresponding Jacobi operator's eigenvalues. In addition, we provide formulas for basis vectors of generalized eigenspaces. These results are of particular interest when questions on diagonalization of non-self-adjoint Jacobi operators are examined.

In Sect. 4, we impose some additional conditions on the diagonal sequence of \mathcal{J} , which allows us to remove singularities of the characteristic function, with possibly one exception located at the origin, and introduce a regularized characteristic function. According to the type of the additional condition, we distinguish 3 different cases and illustrate the respective results on concrete examples. Moreover, in 2 cases concerning either a compact

operator or an operator with compact resolvent, we indicate a connection between the regularized characteristic function and the theory of regularized determinants.

2. The Characteristic Function and Doubly Infinite Jacobi Matrices

In this section, we introduce the characteristic function associated with the doubly infinite Jacobi matrix (1) and derive spectral results similar to those obtained in [15]. Where the verification of a result is completely analogous to the corresponding one given in [15] for the semi-infinite matrix, the proof is only indicated for the sake of brevity.

2.1. Function \mathfrak{F}

The main algebraic tool for the definition of the characteristic function is a function called \mathfrak{F} which appeared in [14] for the first time. The definition given below is a slight generalization of the original one from [14, Def. 1] and is consistent with the one mentioned in [16, Sec. 2].

Definition 1. We define $\mathfrak{F} : \text{Dom } \mathfrak{F} \rightarrow \mathbb{C}$ by the formula

$$\mathfrak{F}(\{x_k\}_{k=-\infty}^{\infty}) := 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=-\infty}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \prod_{j=1}^m x_{k_j} x_{k_j+1}, \quad (2)$$

where

$$\text{Dom } \mathfrak{F} := \left\{ \{x_k\}_{k=-\infty}^{\infty} \subset \mathbb{C} \mid \sum_{k=-\infty}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

Further, if $\{x_k\}_{k=n_1}^{n_2}$, with $n_1, n_2 \in \mathbb{Z} \cup \{\pm\infty\}$, $n_1 \leq n_2$, is given, we put

$$\mathfrak{F}(\{x_k\}_{k=n_1}^{n_2}) := \mathfrak{F}(\{x_k\}_{k=-\infty}^{\infty}),$$

where $x_k := 0$, whenever $k < n_1$ or $k > n_2$, and provided that $\{x_k\}_{k=-\infty}^{\infty} \in \text{Dom } \mathfrak{F}$. Conventionally, for $n_1, n_2 \in \mathbb{Z}$, we also put

$$\mathfrak{F}(\{x_k\}_{k=n_1}^{n_2}) := 1, \text{ if } n_2 = n_1 - 1, \text{ and } \mathfrak{F}(\{x_k\}_{k=n_1}^{n_2}) := 0, \text{ if } n_2 = n_1 - 2.$$

Remark 2. Note that the absolute value of the m th summand on the RHS of (2) is majorized by the expression

$$\sum_{\substack{k \in \mathbb{Z}^m \\ k_1 < k_2 < \cdots < k_m}} \prod_{j=1}^m |x_{k_j} x_{k_j+1}| \leq \frac{1}{m!} \left(\sum_{k=-\infty}^{\infty} |x_k x_{k+1}| \right)^m,$$

cf. [14, Rem. 2]. Consequently, the function \mathfrak{F} is well defined on sequences from $\text{Dom } \mathfrak{F}$, and we have the estimate

$$|\mathfrak{F}(\{x_k\}_{k=-\infty}^{\infty})| \leq \exp \left(\sum_{k=-\infty}^{\infty} |x_k x_{k+1}| \right).$$

Recall several properties of \mathfrak{F} . In the following formulas we always assume that all the expressions are well defined, i.e., the vector in the argument of \mathfrak{F} (possibly with additional zeros) belongs to $\text{Dom } \mathfrak{F}$. First of all, we have the important relation [15, Eq. (19)]

$$\begin{aligned} \mathfrak{F}(\{x_k\}_{k=n_1}^{n_2}) &= \mathfrak{F}(\{x_k\}_{k=n_1}^n) \mathfrak{F}(\{x_k\}_{k=n+1}^{n_2}) \\ &\quad - x_n x_{n+1} \mathfrak{F}(\{x_k\}_{k=n_1}^{n-1}) \mathfrak{F}(\{x_k\}_{k=n+2}^{n_2}), \end{aligned} \quad (3)$$

where $n \in \mathbb{Z}$ satisfies $n_1 \leq n \leq n_2$ and $n_1, n_2 \in \mathbb{Z} \cup \{\pm\infty\}$. Equivalently, (3) can be written as

$$\begin{aligned} \mathfrak{F}(\{x_k\}_{k=n_1}^{n_2}) &= \mathfrak{F}(\{x_k\}_{k=n_1}^n) \mathfrak{F}(\{x_k\}_{k=n+1}^{n_2}) + \mathfrak{F}(\{x_k\}_{k=n_1}^{n-1}) \mathfrak{F}(\{x_k\}_{k=n}^{n_2}) \\ &\quad - \mathfrak{F}(\{x_k\}_{k=n_1}^{n-1}) \mathfrak{F}(\{x_k\}_{k=n+1}^{n_2}), \end{aligned} \quad (4)$$

where we have used that

$$\mathfrak{F}(\{x_k\}_{k=n}^{n_2}) = \mathfrak{F}(\{x_k\}_{k=n+1}^{n_2}) - x_n x_{n+1} \mathfrak{F}(\{x_k\}_{k=n+2}^{n_2}), \quad n \leq n_2, \quad (5)$$

which is the special case of (3) with $n_1 := n$. Similarly, by putting $n_2 := n+1$ in (3), we obtain

$$\mathfrak{F}(\{x_k\}_{k=n_1}^{n+1}) = \mathfrak{F}(\{x_k\}_{k=n_1}^n) - x_n x_{n+1} \mathfrak{F}(\{x_k\}_{k=n_1}^{n-1}), \quad n \geq n_1 - 1. \quad (6)$$

Second, one has the limit relations

$$\lim_{n \rightarrow \infty} \mathfrak{F}(\{x_k\}_{k=n}^{\infty}) = 1 \quad \text{and} \quad \lim_{n \rightarrow -\infty} \mathfrak{F}(\{x_k\}_{k=-\infty}^n) = 1 \quad (7)$$

and

$$\mathfrak{F}(\{x_k\}_{k=-\infty}^{\infty}) = \lim_{n \rightarrow -\infty} \mathfrak{F}(\{x_k\}_{k=n}^{\infty}) = \lim_{n \rightarrow \infty} \mathfrak{F}(\{x_k\}_{k=-\infty}^n),$$

which one verifies in the same way as [15, Lem. 2].

2.2. The Characteristic Function

For two given sequences satisfying a certain convergence condition, we will define a complex function defined on a subset of \mathbb{C} in terms of \mathfrak{F} . This function will be called the characteristic function associated with the doubly infinite Jacobi matrix \mathcal{J} since it plays a similar role to the characteristic polynomial in linear algebra, as it will be further demonstrated.

First, let us introduce a notation we will use. For $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ a complex sequence, we put $\text{Ran}(\lambda) := \{\lambda_n \mid n \in \mathbb{Z}\}$ and $\mathbb{C}_0^\lambda := \mathbb{C} \setminus \overline{\text{Ran}(\lambda)}$, where $\overline{\text{Ran}(\lambda)}$ is the closure of $\text{Ran}(\lambda)$. Further, we denote by $\text{der}(\lambda)$ the set of all (finite) accumulation points of $\text{Ran}(\lambda)$, i.e., $\text{der}(\lambda)$ is the set of all limit points of all possible convergent subsequences of λ . Clearly, $\overline{\text{Ran}(\lambda)} = \text{Ran}(\lambda) \cup \text{der}(\lambda)$.

For $\lambda, w : \mathbb{Z} \rightarrow \mathbb{C}$, the characteristic function will be defined under the condition

$$\sum_{n=-\infty}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty \quad (8)$$

that is valid for at least one $z_0 \in \mathbb{C}_0^\lambda$. If this is true, then (8) remains valid for all $z_0 \in \mathbb{C}_0^\lambda$ and the convergence is locally uniform on \mathbb{C}_0^λ , as it is straightforward to verify; cf. the proof of [15, Lem. 8]. In fact, one also readily shows that (8) remains valid also for $z \in \text{Ran}(\lambda) \setminus \text{der}(\lambda)$ provided the finite number of terms where we would divide by zero are omitted in the series.

Definition 3. Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. We define the *characteristic function associated with the doubly infinite matrix \mathcal{J}* given by (1) by

$$F_{\mathcal{J}}(z) := \mathfrak{F} \left(\left\{ \frac{\gamma_n^2}{z - \lambda_n} \right\}_{n=-\infty}^{\infty} \right), \quad \forall z \in \mathbb{C}_0^\lambda, \quad (9)$$

where $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ is any sequence satisfying the difference equation $\gamma_n \gamma_{n+1} = w_n$ for all $n \in \mathbb{Z}$.

Remark 4. Note that since the sequence in the argument of \mathfrak{F} in (9) fulfills (8) for all $z_0 \in \mathbb{C}_0^\lambda$, it belongs to $\text{Dom } \mathfrak{F}$ and the RHS of (9) is therefore well defined for all $z \in \mathbb{C}_0^\lambda$. Further, since $0 \notin \text{Ran } w$, the sequence γ always exists and is determined uniquely by specifying one value, for example, by setting $\gamma_0 = 1$. The definition of the characteristic function does not depend on a particular choice of the sequence γ .

By a simple modification of the argument used in the proof of [15, Lem. 8], one verifies that

$$\lim_{n \rightarrow \infty} \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-n}^n \right) = F_{\mathcal{J}}(z), \quad (10)$$

and the convergence is locally uniform on \mathbb{C}_0^λ , provided the condition (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. Consequently, $F_{\mathcal{J}}$ is an analytic function on \mathbb{C}_0^λ . Moreover, $F_{\mathcal{J}}$ is meromorphic on $\mathbb{C} \setminus \text{der}(\lambda)$. Indeed, the singularity of $F_{\mathcal{J}}$ at a point $z = \lambda_n$, for some $n \in \mathbb{Z}$ such that $\lambda_n \notin \text{der}(\lambda)$, is either a removable singularity or a pole of order less than or equal to $r(z)$ where

$$r(z) := \#\{n \in \mathbb{Z} \mid \lambda_n = z\} \quad (11)$$

is the number of values of λ coinciding with z . To see this, for $z \in \text{Ran}(\lambda) \setminus \text{der}(\lambda)$ fixed, take $m, M \in \mathbb{Z}$, $m \leq M$, such that $\lambda_n \neq z$ for all $n \leq m$ and all $n \geq M$. Using the rule (4) twice, one derives, for $u \in \mathbb{C}_0^\lambda$, that

$$\begin{aligned} F_{\mathcal{J}}(u) &= (F_{-\infty}^m(u) - F_{-\infty}^{m-1}(u)) \left[F_{m+1}^M(u) F_{M+1}^\infty(u) + F_{m+1}^{M-1}(u) F_M^\infty(u) \right. \\ &\quad \left. - F_{m+1}^{M-1}(u) F_{M+1}^\infty(u) \right] + F_{-\infty}^{m-1}(u) \left[F_m^M(u) F_{M+1}^\infty(u) + F_m^{M-1}(u) F_M^\infty(u) \right. \\ &\quad \left. - F_m^{M-1}(u) F_{M+1}^\infty(u) \right], \end{aligned} \quad (12)$$

where we temporarily denote

$$F_r^s(u) := \mathfrak{F} \left(\left\{ \frac{\gamma_n^2}{u - \lambda_n} \right\}_{n=r}^s \right), \quad r, s \in \mathbb{Z} \cup \{\pm\infty\}, \quad r \leq s,$$

for brevity. Clearly, only functions F_{m+1}^M , F_{m+1}^{M-1} , F_m^M , and F_m^{M-1} can have a singularity at z which can be either a removable singularity or a pole of order at most $r(z)$. The remaining terms on the RHS of (12) are functions analytic at z .

2.3. The Jacobi Operator

Let us recall the standard procedure of prescribing densely defined and closed operators associated with \mathcal{J} . On the algebraic level, the formal doubly infinite matrix \mathcal{J} can be understood as a linear mapping acting on the space of complex sequences x (indexed by \mathbb{Z}) by a formal matrix multiplication, i.e.,

$$(\mathcal{J}x)_n = w_{n-1}x_{n-1} + \lambda_n x_n + w_n x_{n+1}, \quad \forall n \in \mathbb{Z}. \quad (13)$$

Let $\{e_n \mid n \in \mathbb{Z}\}$ stands for the standard basis of $\ell^2(\mathbb{Z})$, i.e., $(e_n)_m = \delta_{m,n}$ for $m, n \in \mathbb{Z}$. Define an auxiliary operator J_0 as

$$J_0 := \mathcal{J} \upharpoonright \text{span}\{e_n \mid n \in \mathbb{Z}\}.$$

The operator J_0 , as an operator on the Hilbert space $\ell^2(\mathbb{Z})$, need not be closed in $\ell^2(\mathbb{Z})$, but is always closable, cf. [3, Subsec. 2.1]. Hence, one can introduce the so-called minimal operator J_{\min} as the operator closure of J_0 . On the other hand, it is natural to define the maximal operator J_{\max} by putting

$$\text{Dom } J_{\max} := \{x \in \ell^2(\mathbb{Z}) \mid \mathcal{J}x \in \ell^2(\mathbb{Z})\}$$

and

$$J_{\max}x := \mathcal{J}x, \quad \text{for } x \in \text{Dom } J_{\max}.$$

Here, the expression $\mathcal{J}x$ is to be understood as in (13).

One can show that $J_{\min} \subset J_{\max}$, but the equality does not hold in general. The operators J_{\min} and J_{\max} are related via their adjoints:

$$J_{\max} = \mathcal{C}J_{\min}^* \mathcal{C} \quad \text{and} \quad J_{\min} = \mathcal{C}J_{\max}^* \mathcal{C}, \quad (14)$$

where \mathcal{C} stands for the complex conjugation operator acting on complex sequences as $(\mathcal{C}x)_n := \overline{x_n}$, $n \in \mathbb{Z}$. The verification of formulas in (14) is a matter of straightforward use of the definition of the adjoint operator; see also [3, Lem. 2.1] for an analogous proof for operators associated with a semi-infinite complex Jacobi matrix.

It follows from (14) that J_{\max} is closed. In addition, any closed linear operator A acting on $\ell^2(\mathbb{Z})$ such that $\{e_n \mid n \in \mathbb{Z}\} \subset \text{Dom } A$, whose matrix representation with respect to the standard basis coincides with \mathcal{J} , satisfies $J_{\min} \subset A \subset J_{\max}$. If $J_{\min} = J_{\max}$, the Jacobi matrix \mathcal{J} determines uniquely the Jacobi operator. In this case, the subscripts can be omitted and we simply write $J := J_{\min} = J_{\max}$.

2.4. Spectral Properties of the Jacobi Operator via the Characteristic Function

For $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$, $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$, and $z \notin \text{Ran}(\lambda)$, we put

$$\mathcal{P}_n(z) := \prod_{k=1}^n \frac{w_{k-1}}{z - \lambda_k}, \quad \text{if } n \geq 0, \quad \text{and} \quad \mathcal{P}_n(z) := \prod_{k=n+1}^0 \frac{z - \lambda_k}{w_{k-1}}, \quad \text{if } n < 0. \quad (15)$$

Note that $\mathcal{P}(z) : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ satisfies the equations

$$\mathcal{P}_0(z) = 1, \quad \text{and} \quad \mathcal{P}_{n+1}(z) = \frac{w_n}{z - \lambda_{n+1}} \mathcal{P}_n(z), \quad \forall n \in \mathbb{Z}. \quad (16)$$

Further, for $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $z \in \mathbb{C}$, we introduce quantities

$$r_+(z) := \#\{n > 0 \mid \lambda_n = z\} \quad \text{and} \quad r_-(z) := \#\{n \leq 0 \mid \lambda_n = z\}. \quad (17)$$

Thus, $r_{\pm}(z) \in \{0, 1, 2, \dots, \infty\}$ and $r_+(z) + r_-(z) = r(z)$ where $r(z)$ is defined in (11). Note that $r_{\pm}(z) < \infty$, if $z \notin \text{der}(\lambda)$, and $r_{\pm}(z) = 0$, if $z \notin \text{Ran}(\lambda)$.

Definition 5. Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. For $z \in \mathbb{C}_0^\lambda$, we define two sequences $f(z), g(z) : \mathbb{Z} \rightarrow \mathbb{C}$ by putting

$$f_n(z) := \mathcal{P}_n(z) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+1}^\infty \right), \quad (18)$$

and

$$g_n(z) := \frac{1}{w_{n-1} \mathcal{P}_{n-1}(z)} \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{n-1} \right), \quad (19)$$

for $n \in \mathbb{Z}$, where $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ is as in Definition 3 and $\mathcal{P}_n(z)$ given by (15). Further, we extend the definition of $f(z)$ and $g(z)$ for $z \in \text{Ran}(\lambda) \setminus \text{der}(\lambda)$ by formulas

$$f_n(z) := \lim_{u \rightarrow z} (u - z)^{r_+(z)} f_n(u) \quad \text{and} \quad g_n(z) := \lim_{u \rightarrow z} (u - z)^{r_-(z)} g_n(u), \quad (20)$$

where $r_{\pm}(z)$ are defined by (17).

Remark 6. Note that the sequences $f(z)$ and $g(z)$ are well defined by Definition 5 for all $z \in \mathbb{C} \setminus \text{der}(\lambda)$. In addition, for $z \in \mathbb{C} \setminus \text{der}(\lambda)$ and $n > 0$ such that $n \geq \max\{k \in \mathbb{Z} \mid \lambda_k = z\}$, one has

$$f_n(z) = \left(\prod_{\substack{k=1 \\ \lambda_k \neq z}}^n \frac{w_{k-1}}{z - \lambda_k} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+1}^\infty \right). \quad (21)$$

Similarly, for $n \leq 0$ such that $n \leq \min\{k \in \mathbb{Z} \mid \lambda_k = z\}$, one has

$$g_n(z) = \frac{1}{w_{n-1}} \left(\prod_{\substack{k=n \\ \lambda_k \neq z}}^0 \frac{w_{k-1}}{z - \lambda_k} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{n-1} \right). \quad (22)$$

Proposition 7. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then $f(z)$ and $g(z)$ are solutions of the eigenvalue equation $\mathcal{J}u = zu$ for all $z \in \mathbb{C} \setminus \text{der}(\lambda)$. In addition, for their Wronskian $W(f(z), g(z)) := w_n (f_n(z)g_{n+1}(z) - f_{n+1}(z)g_n(z))$ (n is arbitrary), one has*

$$W(f(z), g(z)) = \lim_{u \rightarrow z} (u - z)^{r(z)} F_{\mathcal{J}}(u), \quad \forall z \in \mathbb{C} \setminus \text{der}(\lambda).$$

Proof. We verify the statement for $z \in \mathbb{C}_0^\lambda$ only. The extension to all $z \in \mathbb{C} \setminus \text{der}(\lambda)$ is to be treated readily with the aid of limit formulas (20).

By using the definition relations (18), (19) and taking into account the recurrence (16), the verification of the equation $\mathcal{J}u = zu$, i.e.,

$$w_{n-1}u_{n-1} + (\lambda_n - z)u_n + w_n u_{n+1} = 0, \quad \forall n \in \mathbb{Z},$$

for $u = f(z)$ and $u = g(z)$, is a straightforward application of the identities (5) and (6).

Further, for $n \in \mathbb{Z}$ and $z \in \mathbb{C}_0^\lambda$ arbitrary, we have

$$\begin{aligned} W(f(z), g(z)) &= \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^n \right) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+1}^\infty \right) \\ &\quad - \frac{w_n^2}{(z - \lambda_n)(z - \lambda_{n+1})} \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{n-1} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+2}^\infty \right). \end{aligned}$$

According to (3), the RHS of the above equality coincides with $F_{\mathcal{J}}(z)$ which completes the proof. \square

For the spectral analysis of a Jacobi operator associated to \mathcal{J} , we will need to know the asymptotic behavior of the sequences $f(z)$ and $g(z)$ from Definition 5, for $z \in \mathbb{C} \setminus \text{der}(\lambda)$, as the index approaches $+\infty$ and $-\infty$, respectively. The following auxiliary result will be used for this purpose.

Lemma 8. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then*

$$\sum_{n=1}^{\infty} \prod_{\substack{k=1 \\ \lambda_k \neq z}}^n \left| \frac{w_{k-1}}{z - \lambda_k} \right| < \infty \quad \text{and} \quad \sum_{n=-\infty}^0 \frac{1}{|w_{n-1}|} \prod_{\substack{k=n \\ \lambda_k \neq z}}^0 \left| \frac{w_{k-1}}{z - \lambda_k} \right| < \infty,$$

for all $z \in \mathbb{C} \setminus \text{der}(\lambda)$.

Proof. We verify the convergence of the first series. The second one is to be treated similarly.

Let $z \in \mathbb{C} \setminus \text{der}(\lambda)$ be fixed and denote by $M(z) \in \mathbb{N}$ an index such that $z \neq \lambda_n$ for all $n \geq M(z)$. Since

$$\sum_{n=M(z)}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| < \infty$$

one has

$$\lim_{n \rightarrow \infty} \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} = 0$$

and hence, without loss of generality, we may assume that

$$|w_n| \leq \frac{1}{2} |(\lambda_n - z)(\lambda_{n+1} - z)|^{1/2}, \quad \forall n \geq M(z). \quad (23)$$

Next, with the aid of (23), one obtains

$$\prod_{\substack{k=1 \\ \lambda_k \neq z}}^n \left| \frac{w_{k-1}}{z - \lambda_k} \right| \leq \frac{C(z)}{2^n |z - \lambda_n|^{1/2}}, \quad \forall n \geq M(z), \quad (24)$$

where

$$C(z) := 2^{M(z)} |z - \lambda_{M(z)}|^{1/2} \prod_{\substack{k=1 \\ \lambda_k \neq z}}^{M(z)} \left| \frac{w_{k-1}}{z - \lambda_k} \right|.$$

Since $z \neq \lambda_n$ for all $n \geq M(z)$ and $z \notin \text{der}(\lambda)$,

$$|z - \lambda_n| \geq \text{dist} \left(z, \overline{\{\lambda_m \mid m \geq M(z)\}} \right) > 0, \quad \forall n \geq M(z),$$

hence, the RHS of the inequality (24) is the convergent majorant for the first series from the statement. \square

Next, for the purpose of the main theorem of this section, we introduce an extended zero set of the characteristic function $F_{\mathcal{J}}$.

Definition 9. For $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ being such that (8) holds for some $z_0 \in \mathbb{C}_0^\lambda$, we define

$$\mathfrak{Z}(\mathcal{J}) := \left\{ z \in \mathbb{C} \setminus \text{der}(\lambda) \mid \lim_{u \rightarrow z} (u - z)^{r(z)} F_{\mathcal{J}}(z) = 0 \right\}.$$

Remark 10. Note that $\mathfrak{Z}(\mathcal{J})$ decomposes into the union of the set of all zeros of $F_{\mathcal{J}}$ located in \mathbb{C}_0^λ (since $r(z) = 0$ for all $z \in \mathbb{C}_0^\lambda$) and the set of those points from $\text{Ran}(\lambda) \setminus \text{der}(\lambda)$ which are not the poles of $F_{\mathcal{J}}$ of order $r(z)$, i.e., they are either removable singularities or poles of order strictly less than $r(z)$.

Proposition 11. Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then one has

$$\mathfrak{Z}(\mathcal{J}) \subset \text{spec}_p(J_{\max}) \setminus \text{der}(\lambda) \quad (25)$$

and, for $z \in \mathfrak{Z}(\mathcal{J})$, the corresponding eigenvector of J_{\max} can be chosen as $f(z)$.

Proof. For $z \in \mathbb{C} \setminus \text{der}(\lambda)$ fixed, we have, by (7), (21), and (22), that

$$f_n(z) = \alpha_n(z) [1 + o(1)], \quad \text{as } n \rightarrow \infty, \quad (26)$$

and

$$g_n(z) = (w_{n-1} \alpha_{n-1}(z))^{-1} [1 + o(1)], \quad \text{as } n \rightarrow -\infty,$$

where

$$\alpha_n(z) = \prod_{\substack{k=1 \\ \lambda_k \neq z}}^n \frac{w_{k-1}}{z - \lambda_k}, \quad \text{for } n > 0, \quad \text{and} \quad \alpha_n(z) = \prod_{\substack{k=n \\ \lambda_k \neq z}}^0 \frac{z - \lambda_k}{w_{k-1}}, \quad \text{for } n \leq 0.$$

Lemma 8 implies that $\alpha(z)$ is a summable sequence at $+\infty$ and $(w\alpha(z))^{-1}$ is a summable sequence at $-\infty$. Consequently, for all $z \in \mathbb{C} \setminus \text{der}(\lambda)$, $f(z)$ is square summable at $+\infty$ and $g(z)$ square summable at $-\infty$.

Assume $z \in \mathfrak{Z}(\mathcal{J})$. Then, according to Proposition 7, $f(z)$ and $g(z)$ are two solutions of the eigenvalue equation $\mathcal{J}u = zu$, which are linearly dependent since their Wronskian vanishes. Hence, by the above discussion, $f(z)$ and $g(z)$ belong to $\ell^2(\mathbb{Z})$. Particularly, we have $f(z) \in \ell^2(\mathbb{Z})$ and $J_{\max}f(z) = zf(z)$. Thus, if $f(z) \neq 0$, then z is an eigenvalue of J_{\max} and $f(z)$ the corresponding eigenvector. Assume, on the contrary, that $f(z) = 0$. Then, by (21), one has

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=n}^{\infty}\right) = 0,$$

for all $n \in \mathbb{N}$ sufficiently large. However, according to (7), the LHS of the above equality tends to 1, as $n \rightarrow \infty$, which is a contradiction. \square

Next, we derive a summation formula which, for real w and λ , turns out to be the ℓ^2 -norm of an eigenvector of J_{\max} corresponding to an eigenvalue located in \mathbb{C}_0^λ .

Proposition 12. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for some $z_0 \in \mathbb{C}_0^\lambda$ and let $z \in \mathbb{C}_0^\lambda$ be a zero of $F_{\mathcal{J}}$. Then one has*

$$\sum_{n=-\infty}^{\infty} f_n^2(z) = A(z)F'_{\mathcal{J}}(z), \quad (27)$$

where $A(z)$ is given by the formula

$$A(z) = w_{n-1}\mathcal{P}_n(z)\mathcal{P}_{n-1}(z)\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=n+1}^{\infty}\right) / \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=-\infty}^{n-1}\right), \quad (28)$$

with an arbitrary $n \in \mathbb{Z}$ such that the denominator does not vanish.

Proof. According to Proposition 7, $f(z)$ is a solution of the second-order difference equation

$$w_{n-1}u_n + (\lambda_n - z)u_n + w_n u_{n+1} = 0, \quad \forall n \in \mathbb{Z}, \forall z \in \mathbb{C}_0^\lambda.$$

By application of the Green formula, one obtains

$$(x - y) \sum_{k=m+1}^n f_k(x)f_k(y) = W_m(f(x), f(y)) - W_n(f(x), f(y)), \quad (29)$$

for all $m, n \in \mathbb{Z}$, $m \leq n$ and all $x, y \in \mathbb{C}_0^\lambda$, where $W_n(f(x), f(y)) := w_n(f_n(x)f_{n+1}(y) - f_{n+1}(x)f_n(y))$. With the aid of the asymptotic formula (26), one verifies that

$$w_n f_n(x)f_{n+1}(y) = (x - \lambda_0) \left(\prod_{k=0}^n \frac{w_k^2}{(x - \lambda_k)(y - \lambda_{k+1})} \right) (1 + o(1)), \quad (30)$$

for $x, y \in \mathbb{C}_0^\lambda$ and $n \rightarrow \infty$. Note that

$$\left| \frac{w_n^2}{(x - \lambda_n)(y - \lambda_{n+1})} \right| \leq \left| \frac{w_n^2}{(x - \lambda_n)(x - \lambda_{n+1})} \right| \left(1 + \frac{|x - y|}{\text{dist}(y, \text{Ran}(\lambda))} \right).$$

The RHS in the above estimate tends to 0, as $n \rightarrow \infty$, which follows from the assumption (8). Consequently, the product on the RHS of (30) tends to 0, as $n \rightarrow \infty$, and hence

$$\lim_{n \rightarrow \infty} W_n(f(x), f(y)) = 0,$$

for all $x, y \in \mathbb{C}_0^\lambda$. Thus, by sending $n \rightarrow \infty$ in (29), one gets

$$(x - y) \sum_{k=m+1}^{\infty} f_k(x) f_k(y) = W_m(f(x), f(y)), \quad \forall x, y \in \mathbb{C}_0^\lambda. \quad (31)$$

Finally, with the aid of (26) and similarly to what occurs in the proof of Lemma 8, one verifies the sum on the LHS of (31) converges locally uniformly in y on \mathbb{C}_0^λ with $x \in \mathbb{C}_0^\lambda$ fixed (we omit details). Thus, by sending $y \rightarrow x$ in (31) we may interchange the limit and the summation getting

$$\sum_{k=m+1}^{\infty} f_k^2(x) = W_m(f'(x), f(x)), \quad \forall x \in \mathbb{C}_0^\lambda. \quad (32)$$

Analogously, one proves that

$$\sum_{k=-\infty}^n g_k^2(x) = W_n(g(x), g'(x)), \quad \forall x \in \mathbb{C}_0^\lambda. \quad (33)$$

For $z \in \mathbb{C}_0^\lambda$ the zero of $F_{\mathcal{J}}(z)$, the sequences $f(z)$ and $g(z)$ are linearly dependent by Proposition 7. Hence there exists $A(z) \neq 0$ such that

$$f(z) = A(z)g(z). \quad (34)$$

Since $f(z)$ is an eigenvector of J_{\max} by Proposition 11, $A(z) \neq 0$, indeed. By differentiating both sides of the equality $F_{\mathcal{J}}(x) = W(f(x), g(x))$ and making use of (34), one obtains

$$F'_{\mathcal{J}}(z) = A(z)^{-1} W_n(f'(z), f(z)) + A(z) W_n(g(z), g'(z)), \quad \forall n \in \mathbb{Z}.$$

Further, by substituting from (32) and (33) in the above equality, one gets

$$F'_{\mathcal{J}}(z) = A(z)^{-1} \sum_{k=n+1}^{\infty} f_k^2(z) + A(z) \sum_{k=-\infty}^n g_k^2(z), \quad \forall n \in \mathbb{Z}.$$

Finally, by sending $n \rightarrow -\infty$ in the above formula, one arrives at (27). To obtain the expression (28) for $A(z)$, it suffices to use (34) and Definition 5. \square

Remark 13. Note that if the denominator on the RHS of (28) vanishes for some $n \in \mathbb{Z}$, then it is not the case for $n + 1$. Indeed, if

$$\mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^n \right) = \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{n+1} \right) = 0,$$

for some $n \in \mathbb{Z}$, then

$$\mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^n \right) = 0, \quad \forall n \in \mathbb{Z},$$

which one deduces from the recurrence (6). However, this would contradict the second limit relation in (7).

Now we are in position to prove the main result of this section.

Theorem 14. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. Further, assume that $F_{\mathcal{J}}$ does not identically vanish on \mathbb{C}_0^λ . Then it holds:*

- (i) *The matrix \mathcal{J} determines the Jacobi operator uniquely, i.e., $J_{\min} = J_{\max} =: J$.*
- (ii) *One has equalities*

$$\mathfrak{Z}(\mathcal{J}) = \text{spec}_p(J) \setminus \text{der}(\lambda) = \text{spec}(J) \setminus \text{der}(\lambda).$$

- (iii) *The resolvent set $\rho(J)$ of J is nonempty and the Green function*

$$G_{i,j}(z) := \langle e_i, (J - z)^{-1} e_j \rangle, \quad i, j \in \mathbb{Z}, \quad z \in \rho(J),$$

is given by the formula

$$\begin{aligned} G_{i,j}(z) = & -\frac{1}{w_M} \left(\prod_{k=m}^M \frac{w_k}{z - \lambda_k} \right) \\ & \times \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{m-1} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=M+1}^{\infty} \right) / \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{\infty} \right), \end{aligned} \quad (35)$$

for all $z \in \rho(J) \setminus \text{der}(\lambda)$, where $m := \min(i, j)$ and $M := \max(i, j)$. (If $z \in \text{Ran}(\lambda) \setminus \text{der}(\lambda)$, the RHS of (35) is to be understood as the corresponding limit value.)

Proof. We divide the proof into 2 parts.

1) Let $z \notin \mathfrak{Z}(\mathcal{J})$. Such z exists due to the assumption that $F_{\mathcal{J}} \neq 0$ on \mathbb{C}_0^λ . For the sake of simplicity, we will further assume that this z belongs to \mathbb{C}_0^λ . The purpose of this assumption is to avoid complicated expressions caused by the necessary regularization if $z \in \text{Ran}(\lambda) \setminus \text{der}(\lambda)$. However, the idea of the proof remains completely the same.

Let $\mathcal{G}(z)$ be the doubly infinite matrix whose elements are given by the RHS of (35). First, by employing ideas similar to those used in the proof of Lemma 8, one shows that

$$\frac{1}{|w_M|} \left| \prod_{k=m}^M \frac{w_k}{z - \lambda_k} \right| \leq C_1(z) 2^{-|i-j|},$$

for some constant $C_1(z) > 0$ and all $i, j \in \mathbb{Z}$, where $m := \min(i, j)$ and $M := \max(i, j)$. Next, the assumptions also guarantee that the expression

$$\left| \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{m-1} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=M+1}^{\infty} \right) / \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-\infty}^{\infty} \right) \right|$$

is majorized by a constant $C_2(z) > 0$ for all $i, j \in \mathbb{Z}$. Altogether, one has

$$|\mathcal{G}_{i,j}(z)| \leq C(z)2^{-|i-j|}, \quad \forall i, j \in \mathbb{Z}, \quad (36)$$

where the constant $C(z) = C_1(z)C_2(z) > 0$ is independent of the indices i and j .

Now, we may introduce the operator $R(z)$ defined as

$$R(z) := \sum_{s=-\infty}^{\infty} R(z; s), \quad (37)$$

where $R(z; s)$ is the bounded operator acting on $\ell^2(\mathbb{Z})$ determined by its matrix entries $R_{i,j}(z; s) := \delta_{i,j+s} \mathcal{G}_{i,j}(z)$, for $i, j \in \mathbb{Z}$. By (36), the norm of $R(z; s)$ satisfies

$$\|R(z; s)\| = \sup_{i-j=s} |\mathcal{G}_{i,j}(z)| \leq C(z)2^{-|s|}.$$

Consequently, the series (37) converges in the operator norm and $R(z)$ is well defined bounded operator on $\ell^2(\mathbb{Z})$ whose matrix in the standard basis coincides with $\mathcal{G}(z)$.

Note that

$$\mathcal{G}_{i,j}(z) = -\frac{1}{F_{\mathcal{J}}(z)} \begin{cases} f_i(z)g_j(z), & \text{for } i \geq j, \\ f_j(z)g_i(z), & \text{for } i \leq j, \end{cases}$$

where $f(z)$ and $g(z)$ are given in Definition 5. Since, according to Proposition 7, $f(z)$ and $g(z)$ are solutions of the eigenvalue equation $\mathcal{J}u = zu$, one readily verifies that on the level of formal matrix product

$$(\mathcal{J} - z)\mathcal{G}(z) = \mathcal{G}(z)(\mathcal{J} - z) = I. \quad (38)$$

By inspection of domains, the above equalities yield the operators $J_{\max} - z$ and $R(z)$ are mutually inverse and hence

$$R(z) = (J_{\max} - z)^{-1}. \quad (39)$$

Consequently, $z \in \rho(J_{\max})$ and we have shown that $\text{spec}(J_{\max}) \setminus \text{der}(\lambda) \subset \mathfrak{Z}(\mathcal{J})$. Taking also into account (25), we get

$$\mathfrak{Z}(\mathcal{J}) \subset \text{spec}_p(J_{\max}) \setminus \text{der}(\lambda) \subset \text{spec}(J_{\max}) \setminus \text{der}(\lambda) \subset \mathfrak{Z}(\mathcal{J}).$$

Consequently, if we show that the claim (i) holds true, i.e., $J_{\min} = J_{\max}$, the theorem is proved.

2) We show that the assumption of the existence of $z \in \mathbb{C}_0^\lambda$ such that $F_{\mathcal{J}}(z) \neq 0$ implies $J_{\min} = J_{\max}$, indeed. It follows from (14) that

$$\text{Ker}(J_{\max} - z) = \mathcal{C} \text{Ker}((J_{\min} - z)^*) \quad (40)$$

and

$$\text{Ran}(J_{\max} - z) = \mathcal{C} \text{Ran}((J_{\min} - z)^*), \quad (41)$$

for all $z \in \mathbb{C}$.

Let $z \in \mathbb{C}_0^\lambda$ be such that $F_{\mathcal{J}}(z) \neq 0$. Then $z \in \rho(J_{\max})$, as has already been proved in the first part of the proof. Further, by using the second equation in (38), one verifies that

$$R(z)(J_{\min} - z) = I \upharpoonright \text{Dom}(J_{\min}).$$

Hence $J_{\min} - z$ is injective and its left-inverse is $R(z)$. Further, it follows from (41) that $\text{Ran}(J_{\min} - z)$ is a closed subspace, see [9, Chp. IV, Thm. 5.13], and one has

$$\text{Ran}(J_{\min} - z) = [\text{Ker}((J_{\min} - z)^*)]^\perp = \ell^2(\mathbb{Z}).$$

The second equality in the above equation holds since $\text{Ker}((J_{\min} - z)^*) = \{0\}$, which follows from (40) and the injectivity of $J_{\max} - z$. Thus, $J_{\min} - z$ is an invertible operator with a bounded inverse that has to coincide with its left-inverse $R(z)$. Taking into account (39), we obtain

$$(J_{\min} - z)^{-1} = R(z) = (J_{\max} - z)^{-1},$$

which implies, in particular, that $\text{Dom}(J_{\min}) = \text{Dom}(J_{\max})$. \square

Remark 15. Theorem 14 has been derived under two assumptions:

- (i) The convergence condition (8) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$.
- (ii) The function $F_{\mathcal{J}}(z)$ does not vanish identically on \mathbb{C}_0^λ .

The assumption (i) is necessary for the definition of the characteristic function $F_{\mathcal{J}}$ and is essential. The assumption (ii) guarantees that the matrix \mathcal{J} determines the unique Jacobi operator. It might seem hard to decide if the assumption (ii) is fulfilled or not. Let us point out that (ii), if assumed jointly with (i), is not very restrictive and, in applications, it is usually satisfied since $\text{der}(\lambda)$ is typically an empty, 1-point, or 2-point set.

Indeed, (ii) is automatically fulfilled if (i) holds and the $\text{Ran } \lambda$ is contained in a sector $z_0 + \{z \in \mathbb{C} \mid |\arg z - \theta_0| \geq c > 0\}$ for some $z_0 \in \mathbb{C}$ and $\theta_0 \in [0, 2\pi)$. Then the half-line $\mathcal{L} = z_0 + \{re^{i\theta_0} \mid r > 0\}$ is contained in \mathbb{C}_0^λ and $1/|\lambda_k - z|$ tends to 0 monotonically, for all $k \in \mathbb{Z}$, as $z \rightarrow \infty$, $z \in \mathcal{L}$. Similarly as in Remark 2, one derives the estimate

$$|F_{\mathcal{J}}(z) - 1| \leq \exp \left(\sum_{k=-\infty}^{\infty} \left| \frac{w_k^2}{(\lambda_k - z)(\lambda_{k+1} - z)} \right| \right) - 1, \quad \forall z \in \mathbb{C}_0^\lambda.$$

It follows that $F_{\mathcal{J}}(z) \rightarrow 1$ as $z \rightarrow \infty$, $z \in \mathcal{L}$, and hence (ii) is satisfied. Note also that $\text{Ran } \lambda$ is contained in a sector, if it is contained in a half-plane or, in particular, if it is real.

Corollary 16. *Suppose, in addition to the assumptions of Theorem 14, that $|\lambda_n| \rightarrow \infty$ as $n \rightarrow \pm\infty$. Then*

$$\mathfrak{Z}(\mathcal{J}) = \text{spec}(J) = \text{spec}_p(J) \quad (42)$$

and the resolvent $(J - z)^{-1}$ is compact for all $z \in \rho(J)$. Moreover, if there exists $p \geq 1$ such that

$$\sum_{|n| > n_0} \frac{1}{|\lambda_n|^p} < \infty, \quad (43)$$

for some $n_0 \in \mathbb{N}$, then $(J - z)^{-1}$ belongs to the Schatten-von Neumann class \mathcal{S}_p .

Proof. Since $|\lambda_n| \rightarrow \infty$ for $n \rightarrow \pm\infty$, $\text{der}(\lambda) = \emptyset$. Consequently, (42) is a particular case of the statement (ii) of Theorem 14.

Let $z \in \rho(J) = \mathbb{C} \setminus \mathfrak{Z}(\mathcal{J})$ be such that $z \in \mathbb{C}_0^\lambda$. Such z exists since $F_{\mathcal{J}}$ does not vanish identically on \mathbb{C}_0^λ by assumptions. By a slight refinement of the estimate (36), one derives that

$$|\mathcal{G}_{i,j}(z)| \leq \frac{C_2(z)}{|z - \lambda_i|^{1/2} |z - \lambda_j|^{1/2}} 2^{-|i-j|}, \quad \forall i, j \in \mathbb{Z}. \quad (44)$$

This implies that, for $R(z; s)$ defined in (37), $R(z; s)_{i,j} \rightarrow 0$ as $i, j \rightarrow \pm\infty$ with $i - j = s \in \mathbb{Z}$ being fixed. Consequently, $R(z; s)$ is compact for all $s \in \mathbb{Z}$ and, since the series (37) converges in the operator norm, $R(z)$ is also compact.

By assuming (43) additionally, we show that $R(z) \in \mathcal{S}_p$. The operator $R(z; s)(R(z; s))^*$ is a diagonal operator with diagonal elements $|\mathcal{G}_{i,i+s}(z)|^2$, $i \in \mathbb{Z}$. Hence, the numbers $|\mathcal{G}_{i,i+s}(z)|$, $i \in \mathbb{Z}$, are singular values of the compact operator $R(z; s)$. Taking into account (44) and using the Cauchy-Schwarz inequality, one obtains the estimation for the p -th power of the p -th Schatten-von Neumann norm of $R(z; s)$ in the form:

$$\begin{aligned} \|R(z; s)\|_p^p &\leq 2^{-|s|} C_2(z) \sum_{i=-\infty}^{\infty} |z - \lambda_i|^{-p/2} |z - \lambda_{i+s}|^{-p/2} \\ &\leq 2^{-|s|} C_2(z) \sum_{i=-\infty}^{\infty} |z - \lambda_i|^{-p}. \end{aligned}$$

The assumption (43) guarantees the series on the RHS of the above estimate converges. Moreover, it also follows that the series (37) converges in \mathcal{S}_p and hence $R(z) \in \mathcal{S}_p$. \square

3. The Characteristic Function and the Algebraic Multiplicity

The main aim of this section is to prove that the order of a zero of the characteristic function $F_{\mathcal{J}}$ coincides with the algebraic multiplicity of the corresponding eigenvalue. It turns out that the geometric multiplicity of an eigenvalue of the Jacobi operator J under investigation is always one. Consequently, knowledge regarding the algebraic multiplicities of eigenvalues is straightforwardly connected to a possible similarity of J to a diagonal operator. Namely, simplicity of all zeros of $F_{\mathcal{J}}$ is a necessary condition for the possible diagonalizability of J .

3.1. Preliminaries

If A is a closed operator acting on a Hilbert space \mathcal{H} and $z \in \operatorname{spec}_p(A)$, we denote by $\nu_g(z) := \dim \operatorname{Ker}(A - z)$ the geometric multiplicity of A . Further, if z is an isolated point of $\operatorname{spec}(A)$, the algebraic multiplicity of z is defined as $\nu_a(z) := \dim \operatorname{Ran} P_z$, where

$$P_z := -\frac{1}{2\pi i} \oint_{\gamma_z} (A - \xi)^{-1} d\xi$$

is the Riezs spectral projection and γ_z is a positively oriented Jordan curve located in $\rho(A)$ that has z the only spectral point of A in its interior. Recall that $\nu_a(z)$ coincides with the dimension of the space of generalized eigenvectors

$$\mathcal{M} := \{v \in \mathcal{H} \mid (A - z)^n v = 0 \text{ for some } n \in \mathbb{N}\},$$

see, for example, [11, Sec. XII.2].

Let us assume that $J_{\min} = J_{\max} =: J$. Note that $\dim \operatorname{Ker}(J - z)^n \leq 2n$, for any $z \in \mathbb{C}$ and $n \in \mathbb{N}$, since $\operatorname{Ker}(J - z)^n$ is a subspace of the space of solutions of the difference equation $(\mathcal{J} - z)^n u = 0$ which is of order $2n$. Clearly, $\operatorname{Ker}(J - z)^n$ is a subspace of $\operatorname{Ker}(J - z)^{n+1}$ for all $n \in \mathbb{N}$. For $z \in \mathbb{C}$ fixed, let us denote by $\mathcal{M}_1 := \operatorname{Ker}(J - z)$ and by \mathcal{M}_n the orthogonal complement of $\operatorname{Ker}(J - z)^{n-1}$ into $\operatorname{Ker}(J - z)^n$, for $n > 1$. Hence we have

$$\operatorname{Ker}(J - z)^{n+1} = \operatorname{Ker}(J - z)^n \oplus \mathcal{M}_{n+1}, \quad \text{for } n \in \mathbb{N},$$

and

$$\operatorname{Ker}(J - z)^n = \bigoplus_{k=1}^n \mathcal{M}_k \quad \text{and} \quad \mathcal{M} = \bigoplus_{n=1}^{\infty} \mathcal{M}_n.$$

Note that, if $\dim \mathcal{M}_n = 0$ for some $n \in \mathbb{N}$, then $\dim \mathcal{M}_k = 0$ for all $k \geq n$.

Lemma 17. *Let $J_{\min} = J_{\max} =: J$ with $\rho(J) \neq \emptyset$, then $\dim \mathcal{M}_n \leq 1$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$.*

Proof. First, we show that $\dim \mathcal{M}_1 \leq 1$. There is a standard result that goes back to Wall, see [19, Thm. 22.1], which shows that the following claims are equivalent:

- (i) The second-order difference equation $(\mathcal{J} - z)u = 0$ has two linearly independent solutions belonging to $\ell^2(\mathbb{Z})$ for *one* $z \in \mathbb{C}$.
- (ii) The second-order difference equation $(\mathcal{J} - z)u = 0$ has two linearly independent solutions belonging to $\ell^2(\mathbb{Z})$ for *all* $z \in \mathbb{C}$.

Note that $\dim \mathcal{M}_1 \leq 2$. If $\dim \mathcal{M}_1 = 2$, then by the above equivalence, $\dim \operatorname{Ker}(J - z) = 2$ for all $z \in \mathbb{C}$ which contradicts the assumption $\rho(J) \neq \emptyset$. Hence $\dim \mathcal{M}_1 \leq 1$.

Second, we show that $\dim \mathcal{M}_n \leq 1$ for all $n > 1$. For a contradiction, suppose $\dim \mathcal{M}_{n+1} \geq 2$ for some $n \in \mathbb{N}$. Thus, there are two linearly independent vectors $\{f, g\} \subset \mathcal{M}_{n+1}$. One has $(J - z)^n f, (J - z)^n g \in \mathcal{M}_1$ and these vectors are linearly dependent because $\dim \mathcal{M}_1 \leq 1$. Thus, there exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$ or $\beta \neq 0$ and $\alpha(J - z)^n f + \beta(J - z)^n g = 0$. Then $(J - z)^n(\alpha f + \beta g) = 0$, and so $(\alpha f + \beta g) \in \operatorname{Ker}(J - z)^n$. Since $\operatorname{Ker}(J - z)^n$

and \mathcal{M}_{n+1} are mutually orthogonal, we conclude that $\alpha f + \beta g = 0$, which is a contradiction with the linear independence of $\{f, g\}$. \square

Further, we will need the following purely algebraic statement.

Lemma 18. *Let $\{f^{(m)}\}_{m \in \mathbb{N}_0}$ be a sequence of elements of the space of complex sequences such that*

$$\mathcal{J}f^{(0)} = 0 \quad \text{and} \quad \mathcal{J}f^{(m)} = mf^{(m-1)}, \quad \text{for } m \in \mathbb{N}, \quad (45)$$

and $f^{(0)} \neq 0$. Then $\{f^{(m)}\}_{m \in \mathbb{N}_0}$ is linearly independent. In addition, if $\{g^{(m)}\}_{m \in \mathbb{N}_0}$ is another sequence satisfying the equations (45) (with f being replaced by g) and

$$C^{(k)}(f, g) := \sum_{j=0}^k \binom{k}{j} C(f^{(j)}, g^{(k-j)}), \quad (46)$$

where $C_n(f, g) := f_n g_{n+1} - f_{n+1} g_n$, $n \in \mathbb{Z}$, then the implication

$$C^{(\ell)}(f, g) = 0, \quad \forall \ell \in \{0, \dots, k\} \Rightarrow g^{(k)} \in \text{span}\{f^{(0)}, \dots, f^{(k)}\} \quad (47)$$

holds for any $k \in \mathbb{N}_0$.

Proof. First, we verify the linear independence of the set $\{f^{(m)}\}_{m \in \mathbb{N}_0}$. Suppose, for a contradiction, that $\{f^{(m)}\}_{m \in \mathbb{N}_0}$ is linearly dependent. Take $m_0 \in \mathbb{N}$ such that $\{f^{(j)} \mid 1 \leq j \leq m_0\}$ is linearly dependent but $\{f^{(j)} \mid 1 \leq j < m_0\}$ is linearly independent. Such index m_0 exists since $f^{(0)} \neq 0$. Hence there exist numbers $\alpha_0, \alpha_1, \dots, \alpha_{m_0} \in \mathbb{C}$, $\alpha_{m_0} \neq 0$, such that

$$\sum_{j=0}^{m_0} \alpha_j f^{(j)} = 0.$$

By applying \mathcal{J} to both sides of the above equation and using (45), one gets

$$\sum_{j=1}^{m_0} j \alpha_j f^{(j-1)} = 0.$$

The linear independence of $\{f^{(j)} \mid 1 \leq j < m_0\}$ implies $\alpha_j = 0$ for all $j \in \{1, \dots, m_0\}$, which is a contradiction with $\alpha_{m_0} \neq 0$.

Further, the proof of the implication (47) proceeds by the mathematical induction in k . If $k = 0$, then $f^{(0)}$ and $g^{(0)}$ are two solutions of the equation $\mathcal{J}u = 0$ and their Wronskian vanishes. Indeed, for $n \in \mathbb{Z}$ arbitrary, one has $W(f^{(0)}, g^{(0)}) = w_n C_n(f^{(0)}, g^{(0)}) = 0$. Hence, $f^{(0)}$ and $g^{(0)}$ are linearly dependent.

Let the implication (47) hold up to $k - 1$ for some $k \in \mathbb{N}$ and all couples of sequences $\{f^{(m)}\}_{m \in \mathbb{N}_0}$ and $\{g^{(m)}\}_{m \in \mathbb{N}_0}$ satisfying the assumptions of the statement. Suppose that such $\{f^{(m)}\}_{m \in \mathbb{N}_0}$ and $\{g^{(m)}\}_{m \in \mathbb{N}_0}$ are given and

$$C^{(\ell)}(f, g) = 0, \quad \forall \ell \in \{0, \dots, k\}. \quad (48)$$

Consequently, $f^{(0)}$ and $g^{(0)}$ are linearly dependent and hence $g^{(0)} = \alpha f^{(0)}$ for some $\alpha \in \mathbb{C}$. Put

$$\tilde{g}^{(m)} := \frac{1}{m+1} \left(g^{(m+1)} - \alpha f^{(m+1)} \right), \quad \forall m \in \mathbb{N}_0.$$

Since

$$\mathcal{J}\tilde{g}^{(0)} = \mathcal{J}\left(g^{(1)} - \alpha f^{(1)}\right) = g^{(0)} - \alpha f^{(0)} = 0$$

and

$$\mathcal{J}\tilde{g}^{(m)} = \frac{1}{m+1} \mathcal{J}\left(g^{(m+1)} - \alpha f^{(m+1)}\right) = g^{(m)} - \alpha f^{(m)} = m\tilde{g}^{(m-1)},$$

for $m \in \mathbb{N}$, $\{f^{(m)}\}_{m \in \mathbb{N}_0}$ and $\{\tilde{g}^{(m)}\}_{m \in \mathbb{N}_0}$ are a new couple of sequences satisfying the assumptions of the statement. Further, for $\ell \in \mathbb{N}$, one has

$$\begin{aligned} C^{(\ell-1)}(f, \tilde{g}) &= \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} C\left(f^{(\ell-1-j)}, \tilde{g}^{(j)}\right) \\ &= \frac{1}{\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} C\left(f^{(\ell-j)}, g^{(j)} - \alpha f^{(j)}\right). \end{aligned}$$

Taking into account that $C(\cdot, \cdot)$ is a bilinear and antisymmetric form, one obtains

$$C^{(\ell-1)}(f, \tilde{g}) = \frac{1}{\ell} C^{(\ell)}(f, g), \quad \forall \ell \in \mathbb{N}.$$

The above equality together with (48) imply $C^{(\ell)}(f, \tilde{g}) = 0$ for all $\ell \in \{0, \dots, k-1\}$. Consequently, it follows from the induction hypothesis that

$$\tilde{g}^{(k-1)} \in \text{span}\{f^{(0)}, \dots, f^{(k-1)}\}.$$

On the other hand, $\tilde{g}^{(k-1)} = g^{(k)} - \alpha f^{(k)}$ and thus $g^{(k)} \in \text{span}\{f^{(0)}, \dots, f^{(k)}\}$, which concludes the proof. \square

The last preliminary result is a generalization of Lemma 8 with the spectral parameter z restricted to \mathbb{C}_0^λ . It will be used later to show that, for the sequences $f(z)$ and $g(z)$ given in Definition 5, all the derivatives of $f(z)$ are square summable at $+\infty$ and all the derivatives of $g(z)$ are square summable at $-\infty$.

Lemma 19. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then, for all $z \in \mathbb{C}_0^\lambda$ and $k \in \mathbb{N}_0$, one has*

$$\sum_{n=0}^{\infty} \left| \frac{d^k}{dz^k} \mathcal{P}_n(z) \right| < \infty \quad \text{and} \quad \sum_{n=-\infty}^0 \left| \frac{d^k}{dz^k} \frac{1}{w_{n-1} \mathcal{P}_{n-1}(z)} \right| < \infty,$$

where $\mathcal{P}_n(z)$ is given by (15).

Proof. We prove the convergence of the first series only. The verification of the convergence of the second one is analogous.

Let $z \in \mathbb{C}_0^\lambda$. First, the same arguments as in the proof of Lemma 8 show that

$$|\mathcal{P}_n(z)| \leq D(z) 2^{-n}, \tag{49}$$

for all n sufficiently large, where $D(z) > 0$ is a constant independent of n .

Further, note that

$$\mathcal{P}'_n(z) = \xi_n(z) \mathcal{P}_n(z),$$

where

$$\xi_n(z) = \sum_{k=1}^n \frac{1}{\lambda_k - z}.$$

One readily verifies by mathematical induction in $k \in \mathbb{N}$ that the k th derivative of \mathcal{P}_n can be expressed as

$$\mathcal{P}_n^{(k)} = p_k(\xi_n, \xi'_n, \dots, \xi_n^{(k-1)}) \mathcal{P}_n, \quad \text{for } k \in \mathbb{N}, \quad (50)$$

where p_k is a polynomial in $\xi_n, \xi'_n, \dots, \xi_n^{(k-1)}$ of the form

$$p_k(\xi_n, \xi'_n, \dots, \xi_n^{(k-1)}) = \sum_{\alpha \in \mathbb{N}_0^k, |\alpha| \leq k} m_\alpha \prod_{j=1}^k (\xi_n^{(j-1)})^{\alpha_j}, \quad (51)$$

with coefficients $m_\alpha \in \mathbb{Z}$. Since

$$|\xi_n^{(k)}(z)| \leq \frac{k!}{\delta^{k+1}} n, \quad \text{for } k \in \mathbb{N}_0, \quad (52)$$

where $\delta := \text{dist}(z, \text{Ran}(\lambda)) > 0$, one deduces from (50), (51), (52), and (49) that there exists $C_k(z) > 0$ such that

$$|\mathcal{P}_n^{(k)}(z)| \leq C_k(z) n^k 2^{-n},$$

for all n sufficiently large. The above estimate gives a summable majorant for the first series from the statement for arbitrary $k \in \mathbb{N}_0$. \square

3.2. The Multiplicity Theorem

Theorem 20. *Let the assumptions of Theorem 14 be fulfilled. Then the following claims hold true.*

- (i) *For all $z \in \text{spec}_p(J)$, $\nu_g(z) = 1$.*
- (ii) *Suppose additionally that the set $\mathbb{C} \setminus \text{der}(\lambda)$ is connected. Then the set $\text{spec}(J) \cap \mathbb{C}_0^\lambda$ consists of isolated eigenvalues and, if $z \in \mathbb{C}_0^\lambda \cap \text{spec}(J)$, then $\nu_a(z)$ coincides with the order of z as a zero of $F_{\mathcal{J}}$. Moreover, the space of generalized eigenvectors is spanned by vectors $f(z), f'(z), \dots, f^{(\nu_a(z)-1)}(z)$.*

Proof. According to Theorem 14, $J_{\max} = J_{\min} =: J$ and $\rho(J) \neq \emptyset$. Hence the claim (i) follows immediately from Lemma 17.

Next, the fact that $\text{spec}(J) \cap \mathbb{C}_0^\lambda$ contains eigenvalues only follows from the statement (ii) of Theorem 14. If $\mathbb{C} \setminus \text{der}(\lambda)$ is connected, then \mathbb{C}_0^λ is clearly connected as well. Further, to arrive at a contradiction, assume that $\text{spec}(J) \cap \mathbb{C}_0^\lambda$ has an accumulation point. Then by part (ii) of Theorem 14, the set of zeros of the function $F_{\mathcal{J}}$, which is analytic on \mathbb{C}_0^λ , has an accumulation point in \mathbb{C}_0^λ . Thus $F_{\mathcal{J}}$ has to vanish identically on \mathbb{C}_0^λ , a contradiction with the assumption.

Further, let $z_0 \in \mathbb{C}_0^\lambda \cap \text{spec}(J)$, then, by Theorem 14 again, $F_{\mathcal{J}}(z_0) = 0$. Let us denote by n_0 the order of z_0 as the zero of $F_{\mathcal{J}}$. From the formula (35), one observes that any zero of $F_{\mathcal{J}}$ is a pole (or removable singularity) of the Green function of order less or equal to the order of the zero. Thus, any matrix element of the resolvent operator $(J - z)^{-1}$ has a pole at z_0 of order at

most n_0 . Consequently, for any $\phi, \psi \in \ell^2(\mathbb{Z})$, the function $z \mapsto \langle \phi, (J - z)^{-1} \psi \rangle$ has a pole at z_0 of order at most n_0 . It follows from the last assertion that

$$\nu_a(z_0) \leq n_0. \quad (53)$$

Indeed, if $\nu_a(z_0) > n_0$, then there exists a Jordan chain of $J - z_0$ of the length at least $n_0 + 1$, i.e., there are nonzero vectors $\phi_0, \phi_1, \dots, \phi_{n_0} \in \text{Dom } J$ such that

$$(J - z_0)\phi_0 = 0 \quad \text{and} \quad (J - z_0)\phi_k = \phi_{k-1}, \quad \text{for } k = 1, \dots, n_0.$$

From the above equations, one deduces that

$$(J - z)^{-1}\phi_k = \sum_{j=0}^k \frac{1}{(z_0 - z)^{k+1-j}} \phi_j, \quad \text{for } k = 0, \dots, n_0, \quad (54)$$

where z is supposed to be in a neighborhood of z_0 belonging to the resolvent set of J that exists since z_0 is an isolated eigenvalue of J . By putting $k = n_0$ in (54), one observes that $\langle \phi_0, (J - z)^{-1} \phi_{n_0} \rangle$ has a singularity at z_0 of order $n_0 + 1$, a contradiction.

In the remaining part of the proof, we prove that $\{f(z_0), f'(z_0), \dots, f^{(n_0-1)}(z_0)\}$ is a linearly independent set of generalized eigenvectors of J corresponding to the eigenvalue z_0 which, together with (53), will conclude the proof of the assertion (ii).

First, we show that $f^{(j)}(z)$ is a square summable sequence at $+\infty$ for arbitrary $z \in \mathbb{C}_0^\lambda$ and $j \in \mathbb{N}_0$. To this end, it suffices to note that

$$\left| \frac{d^i}{dz^i} \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n}^\infty \right) \right| \leq K(z), \quad (55)$$

for all $n \in \mathbb{N}$ and $0 \leq i \leq j$, with some $K(z) > 0$. This holds true since

$$\lim_{n \rightarrow \infty} \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n}^\infty \right) = 1$$

and the convergence is local uniform in z on \mathbb{C}_0^λ , as one deduces from the inequality

$$\left| \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n}^\infty \right) - 1 \right| \leq \exp \left(\sum_{k=n}^\infty \left| \frac{w_k^2}{(\lambda_k - z)(\lambda_{k+1} - z)} \right| \right) - 1, \quad \forall z \in \mathbb{C}_0^\lambda,$$

which, in its turn, is obtained similarly as the one from Remark 2. Recalling (18) and using (55), one gets

$$|f_n^{(j)}(z)| \leq K(z) \sum_{i=0}^j \binom{j}{i} \left| \mathcal{P}_n^{(i)}(z) \right|, \quad \forall n \in \mathbb{N}.$$

Now, by applying Lemma 19, one concludes that $f^{(j)}(z)$ is a summable and hence also square summable sequence at $+\infty$. Analogously, with the aid of Lemma 19, one verifies that $g^{(j)}(z)$ is a square summable sequence at $-\infty$ for all $z \in \mathbb{C}_0^\lambda$ and $j \in \mathbb{N}_0$.

Further, recall Proposition 7. By differentiating the equation $\mathcal{J}f(z) = zf(z)$ with respect to z , one obtains equalities

$$(\mathcal{J} - z_0)f(z_0) = 0 \quad \text{and} \quad (\mathcal{J} - z_0)f^{(j)}(z_0) = jf^{(j-1)}(z_0), \quad \forall j \in \mathbb{N}. \quad (56)$$

In addition, $f(z_0) \neq 0$ since it is an eigenvector of J . Clearly, the same holds true if f is replaced by g . Thus, the couple of sequences $\{f^{(j)}(z_0)\}_{j \in \mathbb{N}_0}$ and $\{g^{(j)}(z_0)\}_{j \in \mathbb{N}_0}$ fulfills the assumptions of Lemma 18 where \mathcal{J} is replaced by $\mathcal{J} - z_0$. According to this Lemma, the set $\{f(z_0), f'(z_0), \dots, f^{(n_0-1)}(z_0)\}$ is linearly independent. Further, since

$$F_{\mathcal{J}}(z) = W(f(z), g(z)) = w_n C_n(f(z), g(z)), \quad \forall z \in \mathbb{C}_0^\lambda,$$

for $n \in \mathbb{Z}$ arbitrary, one obtains by differentiation that

$$F_{\mathcal{J}}^{(j)}(z) = w_n C_n^{(j)}(f(z), g(z)), \quad \forall j \in \mathbb{N}_0, \quad \forall n \in \mathbb{Z}, \quad \text{and} \quad \forall z \in \mathbb{C}_0^\lambda, \quad (57)$$

where $C^{(j)}(f(z), g(z))$ is as in (46). Since z_0 is a zero of $F_{\mathcal{J}}$ of order n_0 , $F_{\mathcal{J}}^{(j)}(z_0) = 0$ for $0 \leq j < n_0$, and hence, by (57), $C^{(j)}(f(z_0), g(z_0)) = 0$ for $0 \leq j < n_0$. Thus, Lemma 18 yields

$$\text{span} \left\{ f(z_0), f'(z_0), \dots, f^{(n_0-1)}(z_0) \right\} = \text{span} \left\{ g(z_0), g'(z_0), \dots, g^{(n_0-1)}(z_0) \right\}. \quad (58)$$

Since all the sequences from the span on the LHS of (58) are square summable at $+\infty$ and all the sequences from the span on the RHS of (58) are square summable at $-\infty$ one concludes that

$$f^{(j)}(z_0) \in \ell^2(\mathbb{Z}), \quad \forall j \in \{0, 1, \dots, n_0 - 1\}.$$

Finally, by using the equalities from (56), one gets

$$f^{(j)}(z_0) \in \text{Dom } J \quad \text{and} \quad (J - z_0)^{n_0} f^{(j)}(z_0) = 0,$$

for all $j \in \{0, 1, \dots, n_0 - 1\}$. Hence $\{f(z_0), f'(z_0), \dots, f^{(n_0-1)}(z_0)\}$ is a linearly independent set of generalized eigenvectors. \square

Remark 21. Note that, under the assumptions of Theorem 20, any spectral point of J located in \mathbb{C}_0^λ is an isolated eigenvalue whose algebraic multiplicity is finite. Moreover, the proof of Theorem 20 together with Lemma 17 actually show that, for $z_0 \in \mathbb{C}_0^\lambda$ a zero of $F_{\mathcal{J}}$ of order n_0 , it holds

$$\mathcal{M}_j = \text{span} \{ \hat{f}^{(j-1)}(z_0) \}, \quad \text{for } 1 \leq j \leq n_0,$$

and

$$\mathcal{M}_j = \{0\}, \quad \text{for } j > n_0,$$

where $\{\hat{f}(z_0), \hat{f}'(z_0), \dots, \hat{f}^{(n_0-1)}(z_0)\}$ is the set of vectors obtained by the application of the Gram-Schmidt orthogonalization procedure to the set $\{f(z_0), f'(z_0), \dots, f^{(n_0-1)}(z_0)\}$.

The following corollary gives a necessary condition for J to be diagonalizable, i.e., similar to a diagonal operator. The notion of diagonalizability of a non-self-adjoint operator deserves a more detailed explanation. Usually, the operator of the similarity transformation is required to be bounded with

a bounded inverse. This yields nontrivial questions concerning business of the set of eigenvectors. However, these questions are out of the scope of the current paper, and we do not address them here. Let us only mention that the coincidence of algebraic and geometric multiplicity of all eigenvalues is a necessary condition for an operator to be diagonalizable in any reasonable sense.

Corollary 22. *Let the assumptions of Theorem 14 be fulfilled and let $\mathbb{C} \setminus \text{der}(\lambda)$ be connected. If there exists an eigenvalue $z \in \mathbb{C}_0^\lambda$ of J such that the corresponding eigenvector $v(z)$ satisfies*

$$\sum_{n=-\infty}^{\infty} v_n^2(z) = 0,$$

then $\nu_a(z) > \nu_g(z)$.

Proof. By Theorem 20, $\nu_g(z) = 1$. Thus, $v(z) = cf(z)$ for some $c \neq 0$. By applying Theorem 14 and Proposition 12, we obtain $F_{\mathcal{J}}(z) = F'_{\mathcal{J}}(z) = 0$. Consequently, Theorem 20 implies that $\nu_a(z) \geq 2$. \square

Corollary 23. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{R}$ and $w : \mathbb{Z} \rightarrow \mathbb{R} \setminus \{0\}$ be such that (8) holds for at least one $z_0 \in \mathbb{C}_0^\lambda$ and $\text{der}(\lambda) \neq \mathbb{R}$. Then all the zeros of $F_{\mathcal{J}}$ are real and simple.*

Proof. Since $\text{Ran } \lambda \subset \mathbb{R}$, $F_{\mathcal{J}}$ does not vanish identically on \mathbb{C}_0^λ , see Remark 15. Further, according to Theorem 14, $J_{\max} = J_{\min} =: J$. Since w is also assumed to be real, J is self-adjoint.

Next, note that, if $\text{der}(\lambda) \neq \mathbb{R}$, then $\mathbb{C} \setminus \text{der}(\lambda)$ is connected. Let $z_0 \in \mathbb{C}_0^\lambda$ be a zero of $F_{\mathcal{J}}$. By Theorems 14 and 20, z_0 is an isolated eigenvalue of the self-adjoint operator J and therefore $z_0 \in \mathbb{R}$. Moreover, by the self-adjointness of J and the claim (i) of Theorem 20, one has $\nu_a(z_0) = \nu_g(z_0) = 1$. Hence, according to the claim (ii) of Theorem 20, z_0 is a simple zero of $F_{\mathcal{J}}$. \square

4. Diagonals Admitting Global Regularization and Connections with Regularized Determinants

Most of the results obtained within Sects. 2 and 3 have been derived with the spectral parameter restricted to the set \mathbb{C}_0^λ . For example, the zeros of the characteristic function $F_{\mathcal{J}}$ coincide with $\text{spec}_p(J) \cap \mathbb{C}_0^\lambda$ provided that the assumptions of Theorem 14 hold. One can even go further and relate the points from $\text{Ran}(\lambda) \setminus \text{der}(\lambda)$ to $\text{spec}_p(J)$, as has been done in the claim (ii) of Theorem 14. Doing so, one is forced to locally regularize the characteristic function by the limit formula

$$\lim_{u \rightarrow z} (u - z)^{r(z)} F_{\mathcal{J}}(u),$$

which is well defined for all $z \in \mathbb{C} \setminus \text{der}(\lambda)$ since $F_{\mathcal{J}}$ has a pole of finite order at $z \in \text{Ran}(\lambda) \setminus \text{der}(\lambda)$ less than or equal to $r(z) < \infty$ or removable singularity. However, the resulting extended function is no more analytic on the open set where it is defined, i.e., $\mathbb{C} \setminus \text{der}(\lambda)$.

On the other hand, under some additional assumptions concerning the diagonal sequence λ , one can regularize $F_{\mathcal{J}}$ globally by multiplying $F_{\mathcal{J}}$ by a suitable function having zeros at the points from $\text{Ran}(\lambda) \setminus \text{der}(\lambda)$ with respective multiplicities. It turns out that the resulting function is either entire or analytic everywhere except the origin. We distinguish 3 different situations where the diagonal sequence admits the global regularization of $F_{\mathcal{J}}$ and, in each case, we formulate a proposition that combines particular results of Theorems 14 and 20 with the set of treated spectral points extended to either \mathbb{C} , or $\mathbb{C} \setminus \{0\}$. Moreover, we provide an illustrative concrete example to each case and, in two cases, we indicate how the regularized characteristic function is related with the theory of regularized determinants [13].

4.1. The Compact Case

In addition to the assumption (8), let us assume

$$\sum_{n=-\infty}^{\infty} |\lambda_n|^p < \infty, \quad (59)$$

for some $p \in \mathbb{N}$. The condition (59) implies $\lambda_n \rightarrow 0$, as $n \rightarrow \pm\infty$, and hence $\text{der}(\lambda) = \{0\}$. Note also that, assuming (59), the condition (8) holds true for any $z_0 \neq 0$ if and only if $w \in \ell^2(\mathbb{Z})$.

Under the condition (59), we can introduce the functions defined by the Hadamard products

$$\Phi_p^+(z) := \prod_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{z}\right) \exp\left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{\lambda_n}{z}\right)^j\right), \quad (60)$$

$$\Phi_p^-(z) := \prod_{n=-\infty}^0 \left(1 - \frac{\lambda_n}{z}\right) \exp\left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{\lambda_n}{z}\right)^j\right), \quad (61)$$

and

$$\Phi_p(z) := \Phi_p^-(z)\Phi_p^+(z), \quad (62)$$

for all $z \in \mathbb{C} \setminus \{0\}$. Functions (60)–(62) are well defined and analytic on $\mathbb{C} \setminus \{0\}$, see, for example, [5, Chp. 11]. In addition, their zeros are values λ_n , for $n \geq 1$, $n \leq 0$, and $n \in \mathbb{Z}$, respectively. Thus, we can regularize the functions f , g , and $F_{\mathcal{J}}$, defined in Definitions 3 and 5, by introducing functions

$$\tilde{f}(z) := \Phi_p^+(z)f(z), \quad \tilde{g}(z) := \Phi_p^-(z)g(z), \quad \text{and} \quad \tilde{F}_{\mathcal{J}}(z) := \Phi_p(z)F_{\mathcal{J}}(z). \quad (63)$$

All the functions \tilde{f} , \tilde{g} , and $\tilde{F}_{\mathcal{J}}$ are analytic on $\mathbb{C} \setminus \{0\}$.

Now, we can formulate a proposition summarizing particular results from Theorem 14 and 20 adjusted to the Jacobi operator whose diagonal sequence satisfies (59). While the first part of the statement is an immediate consequence of Theorem 14, the second part concerning algebraic multiplicities does not follow from Theorem 20 readily. However, to verify this statement, one follows the same steps as in the proof of Theorem 20 with the

functions f and g being replaced by their regularized extensions \tilde{f} and \tilde{g} . For this reason and the sake of brevity, the proof is only indicated.

Proposition 24. *Let $\lambda : \mathbb{Z} \rightarrow \mathbb{C}$ and $w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that $w \in \ell^2(\mathbb{Z})$ and $\lambda \in \ell^p(\mathbb{Z})$, for some $p \in \mathbb{N}$. Then*

$$\operatorname{spec}(J) = \operatorname{spec}_p(J) \cup \{0\} = \{z \in \mathbb{C} \setminus \{0\} \mid \tilde{F}_{\mathcal{J}}(z) = 0\} \cup \{0\}. \quad (64)$$

In addition, the algebraic multiplicity $\nu_a(z)$ of a nonzero eigenvalue z of J coincides with the order of z as a zero of $\tilde{F}_{\mathcal{J}}$ and the space of generalized eigenvectors is spanned by vectors $\tilde{f}(z), \tilde{f}'(z), \dots, \tilde{f}^{(\nu_a(z)-1)}(z)$.

Proof. The assumptions imply that $\operatorname{der}(\lambda) = \{0\}$ and the condition (8) is satisfied for any $z_0 \neq 0$. Moreover, by Remark 15, $F_{\mathcal{J}} \neq 0$ on \mathbb{C}_0^λ since $\operatorname{Ran}(\lambda)$ is bounded. Then Theorem 14 tells us that $J_{\min} = J_{\max} =: J$ and

$$\operatorname{spec}(J) \setminus \{0\} = \operatorname{spec}_p(J) \setminus \{0\} = \{z \in \mathbb{C} \setminus \{0\} \mid \tilde{F}_{\mathcal{J}}(z) = 0\}.$$

To verify (64), it suffices to realize that J is compact and hence $0 \in \operatorname{spec}(J)$. Indeed, one easily shows that the Jacobi operator J is compact if and only if

$$\lim_{n \rightarrow \pm\infty} \lambda_n = \lim_{n \rightarrow \pm\infty} w_n = 0.$$

The above equalities are guaranteed by the assumptions $w \in \ell^2(\mathbb{Z})$ and $\lambda \in \ell^p(\mathbb{Z})$.

The remaining part of the statement is to be derived by the same way as Theorem 14 where the solutions f and g are replaced by their regularized extensions \tilde{f} and \tilde{g} . \square

Let us remark that although $0 \in \operatorname{spec}(J)$, 0 need not be an eigenvalue of J . Note also that the matrix elements of the resolvent, see Theorem 14 part (iii), can be now written as

$$\mathcal{G}_{i,j}(z) = -\frac{1}{\tilde{F}_{\mathcal{J}}(z)} \begin{cases} \tilde{f}_i(z)\tilde{g}_j(z), & \text{for } i \geq j, \\ \tilde{f}_j(z)\tilde{g}_i(z), & \text{for } i \leq j, \end{cases} \quad (65)$$

for all $z \in \rho(J)$. Further, following the same steps as in the proof of Proposition 12 replacing everywhere $f(z)$, $g(z)$, and $F_{\mathcal{J}}(z)$ by $\tilde{f}(z)$, $\tilde{g}(z)$, and $\tilde{F}_{\mathcal{J}}(z)$, respectively, one arrives at the summation formula

$$\sum_{n=-\infty}^{\infty} \tilde{f}_n^2(z) = \tilde{A}(z)\tilde{F}'_{\mathcal{J}}(z), \quad (66)$$

for any $z \neq 0$ such that $\tilde{F}_{\mathcal{J}}(z) = 0$, where $\tilde{A}(z) = \tilde{f}_n(z)/\tilde{g}_n(z)$ for any $n \in \mathbb{Z}$ such that $\tilde{g}_n(z) \neq 0$.

Remark 25. There is a close connection between the regularized characteristic function $\tilde{F}_{\mathcal{J}}$ and the theory of regularized determinants, see [13, Chp. 9]. First, note that J can be decomposed as $J = \Lambda + UW + WU^*$, where $\Lambda e_n = \lambda_n e_n$, $W e_n = w_n e_n$, and $U e_n = e_{n+1}$ for all $n \in \mathbb{Z}$. The assumptions $\lambda \in \ell^p(\mathbb{Z})$ and $w \in \ell^2(\mathbb{Z})$ imply that $\Lambda \in \mathcal{S}_p$ and $W \in \mathcal{S}_2$. Consequently, $J \in \mathcal{S}_p + \mathcal{S}_2 \subset \mathcal{S}_{\max(2,p)}$ since $\mathcal{S}_p \subset \mathcal{S}_q$ for $1 \leq p \leq q \leq \infty$. Thus, if $p \geq 2$, the

regularized determinant $\det_p(1 - zJ)$ is well defined and is an entire function of z . We will show that

$$\tilde{F}_{\mathcal{J}}(z) = \det_p(1 - z^{-1}J), \quad \forall z \in \mathbb{C} \setminus \{0\}. \quad (67)$$

Let P_N stand for the orthogonal projection on the space $\text{span}\{e_n \mid |n| \leq N\}$. Without loss of generality, we may assume that (59) holds with $p \geq 2$. Then, by the above discussion, $J \in \mathcal{S}_p$ and one has $P_N J P_N \rightarrow J$ in \mathcal{S}_p , as $N \rightarrow \infty$. Further, with the aid of the formula for the determinant of a tridiagonal matrix [15, Eq. (13)] and [13, Thm. 9.2(d)] one obtains

$$\begin{aligned} \det_p(1 - zP_N J P_N) \\ = \left[\prod_{n=-N}^N (1 - z\lambda_n) \exp \left(\sum_{j=1}^{p-1} \frac{z^j \lambda_n^j}{j} \right) \right] \mathfrak{F} \left(\left\{ \frac{z\gamma_k^2}{1 - z\lambda_k} \right\}_{k=-N}^N \right). \end{aligned}$$

Now, it suffices to send $N \rightarrow \infty$ in the above formula to verify (67), where one has to take into account that $\det_p(1 + \cdot)$ is a continuous functional on \mathcal{S}_p , see [13, Thm. 9.2(c)], and the formula (10). Having the formula (67) at hand, claims of Proposition 24, with the exception of the one about vectors spanning the generalized eigenspace, may be deduced from general results of the theory of regularized determinants [13].

Finally, we illustrate the results derived within this subsection on a concrete example. We follow the standard notation for hypergeometric series, the Bessel function of the first kind, the gamma, and digamma function as it is used, for example, in [1].

Example 26. For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \{0\}$, put

$$\lambda_n := \frac{1}{n + \alpha} \quad \text{and} \quad w_n := \frac{\beta}{\sqrt{(n + \alpha)(n + 1 + \alpha)}},$$

where $n \in \mathbb{Z}$. Then the assumptions of Proposition 24 hold with $p = 2$. The sequence $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ satisfying $\gamma_n \gamma_{n+1} = w_n$, for all $n \in \mathbb{Z}$, can be chosen as

$$\gamma_n = \sqrt{\frac{\beta}{n + \alpha}}, \quad \text{for } n \in \mathbb{Z},$$

and, for $z \notin \text{Ran}(\lambda) \cup \{0\}$, we have

$$\begin{aligned} \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{z - \lambda_k} \right\}_{k=n+1}^{\infty} \right) &= \mathfrak{F} \left(\left\{ \frac{\beta}{1 - z(k + \alpha)} \right\}_{k=n+1}^{\infty} \right) \\ &= {}_0F_1 \left(-; n + 1 + \alpha - z^{-1}, -\beta^2 z^{-2} \right). \end{aligned} \quad (68)$$

The second equality in (68) follows from [1, Eq. 9.1.69]

$${}_0F_1 \left(-; \nu + 1, -z^2 \right) = \Gamma(\nu + 1) z^{-\nu} J_{\nu}(2z), \quad z \in \mathbb{C}, \quad \nu \notin -\mathbb{N}, \quad (69)$$

and the identity

$$J_{\nu}(2z) = \frac{z^{\nu}}{\Gamma(\nu + 1)} \mathfrak{F} \left(\left\{ \frac{z}{\nu + k} \right\}_{k=1}^{\infty} \right), \quad z \in \mathbb{C}, \quad \nu \notin -\mathbb{N}, \quad (70)$$

which was proved in [14, Ex. 11]. It is easy to see, using the definition of the hypergeometric series ${}_0F_1$, that the RHS of (68) tends to 1 as $n \rightarrow -\infty$. Hence, we have

$$F_{\mathcal{J}}(z) = 1, \quad \forall z \notin \text{Ran}(\lambda) \cup \{0\}. \quad (71)$$

It follows immediately from (71) and Theorem 14 that $\text{Ran } \lambda \subset \text{spec}_p(J)$. Nevertheless, we compute the regularized characteristic function and hence evaluate the Hilbert–Schmidt determinant (67). By definition (60), we have

$$\Phi_2^+(z) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{z(n+\alpha)}\right) \exp\left(\frac{1}{z(n+\alpha)}\right), \quad z \neq 0.$$

Note that the zeros of the entire function $\Phi_2^+(1/z)$ coincide with those of $1/\Gamma(\alpha+1-z)$ which is an entire function of order 1. Consequently, by applying the Hadamard factorization theorem, we obtain

$$\frac{1}{\Gamma(\alpha+1-z)} = e^{a+bz} \Phi_2^+\left(\frac{1}{z}\right) = e^{a+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n+\alpha}\right) e^{z/(n+\alpha)},$$

for some $a, b \in \mathbb{C}$. By putting $z = 0$ in the above formula, one gets $e^a = 1/\Gamma(\alpha+1)$, while, by equating coefficients at z on both sides, one computes $b = \psi(\alpha+1) = \Gamma'(\alpha+1)/\Gamma(\alpha+1)$. Thus, we have

$$\Phi_2^+(z) = e^{-z^{-1}\psi(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-z^{-1})}, \quad z \neq 0. \quad (72)$$

With the aid of (72), one further derives

$$\Phi_2^-(z) = \prod_{n=0}^{\infty} \left(1 - \frac{1}{z(\alpha-n)}\right) \exp\left(\frac{1}{z(\alpha-n)}\right) = e^{z^{-1}\psi(-\alpha)} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha+z^{-1})}, \quad (73)$$

for $z \neq 0$. At last, by multiplying (72) and (73) and using the identities [1, Eqs. 6.1.17, 6.3.7]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{and} \quad \psi(1-z) - \psi(z) = \pi \cot \pi z, \quad z \in \mathbb{C} \setminus \mathbb{Z}, \quad (74)$$

we obtain the regularized characteristic function in the form

$$\tilde{F}_{\mathcal{J}}(z) = \Phi_2(z)F_{\mathcal{J}}(z) = \frac{\sin \pi(\alpha - z^{-1})}{\sin \pi \alpha} e^{z^{-1}\pi \cot \pi \alpha}, \quad \forall z \neq 0, \quad (75)$$

where (71) was used. Equivalently, according to (67), we proved that

$$\det_2(1 - zJ) = \frac{\sin \pi(\alpha - z)}{\sin \pi \alpha} e^{z\pi \cot \pi \alpha}, \quad \forall z \in \mathbb{C}.$$

Having (75), Proposition 24 tells us that

$$\text{spec}(J) = \text{spec}_p(J) \cup \{0\} = \text{Ran } \lambda \cup \{0\} = \{1/(\alpha + n) \mid n \in \mathbb{Z}\} \cup \{0\},$$

and all the nonzero eigenvalues have the algebraic multiplicity equal to 1. Further, by using (68) and (72), we readily compute

$$\begin{aligned}\tilde{f}_n(z) &= \left(\frac{\beta}{z}\right)^n \sqrt{\frac{\alpha+n}{\alpha}} e^{-z^{-1}\psi(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+n-z^{-1})} \\ &\quad \times {}_0F_1(-; n+1+\alpha-z^{-1}, -\beta^2 z^{-2}),\end{aligned}$$

which, with the aid of (69), can be rewritten in terms of Bessel functions as

$$\tilde{f}_n(z) = \left(\frac{\beta}{z}\right)^{-\alpha+z^{-1}} \Gamma(\alpha+1) e^{-z^{-1}\psi(\alpha+1)} \sqrt{\frac{\alpha+n}{\alpha}} J_{n-\alpha-z^{-1}}\left(\frac{2\beta}{z}\right), \quad \forall z \neq 0.$$

Further, if we put $z = z_N := 1/(N + \alpha)$, for $N \in \mathbb{Z}$, in the above formula and omit the factor not depending on n , we obtain the n th entry of the eigenvector corresponding to the eigenvalue z_N in the form

$$v_n(z_N) = \sqrt{\alpha+n} J_{n-N}(2\beta(N+\alpha)), \quad n, N \in \mathbb{Z}.$$

The method based on the characteristic function does not tell us whether the points from $\text{der}(\lambda)$ are spectral points of J or not. In this example, we know that $0 \in \text{spec}(J)$ since J is compact. On the other hand, 0 is not an eigenvalue of J . Indeed, since $w_n = \beta\sqrt{\lambda_n\lambda_{n+1}}$ for all $n \in \mathbb{Z}$, the solution of the eigenvalue equation $Ju = 0$ can be obtained by solving the second-order difference equation with constant coefficients:

$$\beta v_{n-1} + v_n + \beta v_{n+1} = 0, \quad n \in \mathbb{Z},$$

where $v_n := \sqrt{\lambda_n} u_n$. By inspection of the asymptotic behavior of the solution v_n for $n \rightarrow \pm\infty$, one concludes that there is no nontrivial square summable solution of $Ju = 0$ for all $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \{0\}$.

Without going into details, let us also remark that

$$\begin{aligned}\tilde{g}_n(z) &= (-1)^{n-1} \beta^{\alpha-z^{-1}} z^{-\alpha-1+z^{-1}} \Gamma(-\alpha) e^{z^{-1}\psi(-\alpha)} \sqrt{\alpha(\alpha+n)} \\ &\quad \times J_{-n-\alpha+z^{-1}}\left(\frac{2\beta}{z}\right),\end{aligned}$$

for $z \neq 0$. With the aid of identity [1, Eq. 9.1.5]

$$J_{-n}(z) = (-1)^n J_n(z), \quad \forall n \in \mathbb{Z}, \quad (76)$$

one further verifies that

$$\tilde{A}(z_N) = \frac{\tilde{f}_n(z_N)}{\tilde{g}_n(z_N)} = (-1)^N \beta^{2N} z_N^{-2N+1} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} e^{-z_N^{-1}(\psi(\alpha+1)+\psi(-\alpha))}.$$

In addition, one has

$$F'_{\mathcal{J}}(z_N) = (-1)^N \frac{\pi}{\sin \pi \alpha} z_N^{-2} e^{z_N^{-1}(\psi(-\alpha)-\psi(\alpha+1))}.$$

So we can substitute for all the functions into the formula (66) which, after some simplifications, leads to the identity

$$\sum_{n=-\infty}^{\infty} (\alpha+n) J_{n-N}^2(2\beta(N+\alpha)) = N + \alpha, \quad (77)$$

for $\alpha \notin \mathbb{Z}$ and $\beta \neq 0$. Since

$$\sum_{n=-\infty}^{\infty} n J_n^2(z) = 0, \quad \forall z \in \mathbb{C},$$

as one deduces with the aid of (76), the identity (77) yields the well-known summation formula for Bessel functions [1, Eq. 9.1.76]

$$\sum_{n=-\infty}^{\infty} J_n^2(z) = 1,$$

which holds true for all $z \in \mathbb{C}$.

At last, one can make use of (65) to deduce that, for $i \geq j$, the Green function reads

$$\mathcal{G}_{i,j}(z) = \frac{(-1)^{j+1} \pi \sqrt{(i+\alpha)(j+\alpha)}}{z \sin \pi(\alpha - z^{-1})} J_{i+\alpha-z^{-1}} \left(\frac{2\beta}{z} \right) J_{-j-\alpha+z^{-1}} \left(\frac{2\beta}{z} \right),$$

where $z \notin 1/(\mathbb{Z} + \alpha) \cup \{0\}$.

4.2. The Case of Compact Resolvent

In this subsection, we impose an additional assumption to the diagonal sequence of \mathcal{J} by requiring

$$0 \notin \text{Ran}(\lambda) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{1}{|\lambda_n|^p} < \infty, \quad (78)$$

for some $p \in \mathbb{N}$. Clearly, (78) implies $|\lambda_n| \rightarrow \infty$, as $n \rightarrow \pm\infty$, and so $\text{der}(\lambda) = \emptyset$. Since $0 \in \mathbb{C}_0^\lambda$ the condition (8) can be replaced by

$$\sum_{n=-\infty}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty.$$

Note also that the assumption $0 \notin \text{Ran}(\lambda)$ is not very restrictive since one can always shift the diagonal sequence λ by a constant that causes only a shift in the spectral parameter.

Under the condition (78), the functions

$$\Psi_p^+(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n} \right) \exp \left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{z}{\lambda_n} \right)^j \right), \quad (79)$$

$$\Psi_p^-(z) := \prod_{n=-\infty}^0 \left(1 - \frac{z}{\lambda_n} \right) \exp \left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{z}{\lambda_n} \right)^j \right), \quad (80)$$

and

$$\Psi_p(z) := \Psi_p^-(z) \Psi_p^+(z), \quad (81)$$

are well defined and entire, see [5, Chp. 11]. Thus, to regularize the key functions we put

$$\tilde{f}(z) := \Psi_p^+(z) f(z), \quad \tilde{g}(z) := \Psi_p^-(z) g(z), \quad \text{and} \quad \tilde{F}_{\mathcal{J}}(z) := \Psi_p(z) F_{\mathcal{J}}(z),$$

for all $z \in \mathbb{C}$. With some abuse of notation, we use the same notation as in (63), although the functions f , g and $F_{\mathcal{J}}$ are regularized by different functions (79)–(81). This should not cause any confusion since we always work with these regularized (tilde) versions of the functions f , g and $F_{\mathcal{J}}$ within the subsection where they are defined exclusively.

Proposition 27. *Let $\lambda, w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that*

$$\sum_{n=-\infty}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{1}{|\lambda_n|^p} < \infty, \quad \text{for some } p \geq 1.$$

Further, let $\tilde{F}_{\mathcal{J}}$ do not vanish identically on \mathbb{C} . Then $J_{\min} = J_{\max} =: J$ and

$$\text{spec}(J) = \text{spec}_p(J) = \{z \in \mathbb{C} \mid \tilde{F}_{\mathcal{J}}(z) = 0\}.$$

In addition, the algebraic multiplicity $\nu_a(z)$ of an eigenvalue z of J coincides with the order of z as a zero of $\tilde{F}_{\mathcal{J}}$ and the space of generalized eigenvectors is spanned by the vectors $\tilde{f}(z), \tilde{f}'(z), \dots, \tilde{f}^{(\nu_a(z)-1)}(z)$.

Proof. The first two claims of the statement follow readily from Corollary 16. The part on the algebraic multiplicity and the vectors spanning the space of generalized eigenvectors is to be deduced in the similar way as in Theorem 20 with functions f and g being replaced by their regularized extensions \tilde{f} and \tilde{g} . \square

Under the assumptions of Proposition 27, one also has the summation formula for squares of the elements of eigenvectors as well as the formula for the Green function which are of the same form as in (66) and (65).

Remark 28. Similarly as in Remark 25, there is a connection between $\tilde{F}_{\mathcal{J}}$ and regularized determinants. However, this connection is not that straightforward as in the case of compact J .

Under the assumptions of Proposition 27, the auxiliary Jacobi operator $A(z)$, determined by

$$A(z)e_n := \frac{w_{n-1}}{\sqrt{\lambda_{n-1}\lambda_n}}e_{n-1} - \frac{z}{\lambda_n}e_n + \frac{w_n}{\sqrt{\lambda_n\lambda_{n+1}}}e_{n+1}, \quad n \in \mathbb{Z},$$

is bounded for all $z \in \mathbb{C}$. In fact, the assumptions of Proposition 27 imply $A(z) \in \mathcal{S}_p + \mathcal{S}_2 \subset \mathcal{S}_{\max(2,p)}$, for all $z \in \mathbb{C}$. Hence, assuming without loss of generality that $p \geq 2$, $A(z) \in \mathcal{S}_p$ for all $z \in \mathbb{C}$. Since $J_{\min} = J_{\max} =: J$, the operator J coincides with the closure of the operator sum $\Lambda + UW + WU^*$ where Λ , W and U are defined in Remark 25. Thus formally (not taking care about domains), one has

$$J - z = \Lambda^{1/2}(1 + A(z))\Lambda^{1/2}, \quad \forall z \in \mathbb{C}.$$

Consequently, one expects that $z \in \rho(J)$ if and only if $-1 \in \rho(A(z))$. However, this equivalence is true, indeed, since, by using Definition 3, one verifies that $F_{\mathcal{J}}(z) = F_{A(z)}(-1)$ for all $z \notin \text{Ran}(\lambda)$. The rest then follows from the claim (ii) of Theorem 14.

Since the matrix of $A(z)$ is tridiagonal, one can make use of the formula [15, Eq. (13)] to show that

$$\det(1 + P_N A(z) P_N) = \left[\prod_{n=-N}^N \left(1 - \frac{z}{\lambda_n}\right) \right] \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-N}^N \right),$$

where P_N is the projection on $\text{span}\{e_n \mid |n| \leq N\}$. Further, using [13, Thm. 9.2(d)] one obtains

$$\begin{aligned} \det_p(1 + P_N A(z) P_N) \\ = \left[\prod_{n=-N}^N \left(1 - \frac{z}{\lambda_n}\right) \exp \left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{z}{\lambda_n} \right)^j \right) \right] \mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=-N}^N \right). \end{aligned}$$

Finally, by sending $N \rightarrow \infty$ and taking into account [13, Thm. 9.2(c)] together with the formula (10), one arrives at the identity

$$\det_p(1 + A(z)) = \tilde{F}_{\mathcal{J}}(z), \quad \forall z \in \mathbb{C}.$$

Probably the most simple nontrivial example illustrating the situation treated by Proposition 27 is the following one.

Example 29. Put $\lambda_n := n$ and $w_n := w \in \mathbb{C} \setminus \{0\}$ for all $n \in \mathbb{Z}$. Then one can take $\gamma_n = \sqrt{w}$ and, for $n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$, one has

$$\mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{z - \lambda_k} \right\}_{k=n}^{\infty} \right) = \mathfrak{F} \left(\left\{ \frac{w}{k - z} \right\}_{k=n}^{\infty} \right) = {}_0F_1(-; n - z, -w^2), \quad (82)$$

as it follows from (70) and (69). By sending $n \rightarrow -\infty$ in (82), one gets

$$F_{\mathcal{J}}(z) = 1, \quad \forall z \in \mathbb{C} \setminus \mathbb{Z}. \quad (83)$$

Note that the assumptions of Proposition 27 are satisfied with $p = 2$ (see also Remark 15) with the only exception that $0 \in \text{Ran}(\lambda)$. However, this is not an essential obstacle and it can be easily overcome by taking as the regularizing functions

$$\begin{aligned} \Psi_2^+(z) &:= \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} = \frac{e^{\gamma z}}{\Gamma(1 - z)}, \\ \Psi_2^-(z) &:= z \prod_{n=-\infty}^{-1} \left(1 - \frac{z}{n}\right) e^{z/n} = \frac{e^{-\gamma z}}{\Gamma(z)}, \end{aligned}$$

and

$$\Psi_2(z) := \Psi_2^-(z) \Psi_2^+(z) = \frac{1}{\Gamma(1 - z) \Gamma(z)} = \frac{\sin \pi z}{\pi},$$

defined for all $z \in \mathbb{C}$. In the above equalities, we have used the first identity from (74) and the well-known Hadamard product formula for the entire function $1/\Gamma(z)$ involving Euler's constant γ , see [1, Eq. 6.1.3]. With this choice of regularizing functions, Proposition 27 remains valid in the same form. Taking into account (83), the regularized characteristic function reads

$$\tilde{F}_{\mathcal{J}}(z) = \Psi_2(z) = \frac{\sin \pi z}{\pi}, \quad \forall z \in \mathbb{C}.$$

Consequently, $\text{spec}(J) = \text{spec}_p(J) = \mathbb{Z}$ and all the eigenvalues have the algebraic multiplicity equal to 1. Further,

$$\tilde{f}_n(z) = \Psi_2^+(z)f_n(z) = (-1)^n e^{\gamma z} w^z J_{n-z}(2w), \quad \forall z \in \mathbb{C},$$

and hence the n th component of the eigenvector corresponding to the eigenvalue $N \in \mathbb{Z}$ of J can be chosen as

$$v_n(N) = (-1)^n J_{n-N}(2w).$$

4.3. The Combined Case

Finally, we discuss a situation which is a combination of the previous two cases. We suppose that there exists $p \geq 1$ such that

$$\sum_{n=1}^{\infty} |\lambda_n|^p < \infty \quad \text{and} \quad \sum_{n=-\infty}^0 \frac{1}{|\lambda_n|^p} < \infty, \quad (84)$$

where it is assumed that $\lambda_n \neq 0$ for $n \leq 0$. Clearly, $\text{der}(\lambda) = \{0\}$. One can show that, under the assumption (84), the condition (8) holds if and only if

$$\sum_{n=1}^{\infty} |w_n|^2 < \infty \quad \text{and} \quad \sum_{n=-\infty}^0 \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty.$$

The way to regularize f , g , and $F_{\mathcal{J}}$ is now the following:

$$\tilde{f}(z) := \Phi_p^+(z)f(z), \quad \tilde{g}(z) := \Psi_p^-(z)g(z),$$

and

$$\tilde{F}_{\mathcal{J}}(z) := \Phi_p^+(z)\Psi_p^-(z)F_{\mathcal{J}}(z),$$

for $z \in \mathbb{C} \setminus \{0\}$, where Φ_p^+ and Ψ_p^- are given by (60) and (80), respectively.

Since there is no significant difference in the derivation of the following statement in comparison with Propositions 24 and 27, we omit the proof completely.

Proposition 30. *Let $\lambda, w : \mathbb{Z} \rightarrow \mathbb{C} \setminus \{0\}$ be such that*

$$\sum_{n=1}^{\infty} |\lambda_n|^p < \infty \quad \text{and} \quad \sum_{n=-\infty}^0 \frac{1}{|\lambda_n|^p} < \infty, \quad \text{for some } p \geq 1,$$

and

$$\sum_{n=1}^{\infty} |w_n|^2 < \infty \quad \text{and} \quad \sum_{n=-\infty}^0 \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty.$$

Further, let $\tilde{F}_{\mathcal{J}}$ does not vanish identically on \mathbb{C} . Then $J_{\min} = J_{\max} =: J$ and

$$\text{spec}(J) \setminus \{0\} = \text{spec}_p(J) \setminus \{0\} = \{z \in \mathbb{C} \setminus \{0\} \mid \tilde{F}_{\mathcal{J}}(z) = 0\}.$$

In addition, the algebraic multiplicity $\nu_a(z)$ of a nonzero eigenvalue z of J coincides with the order of z as a zero of $\tilde{F}_{\mathcal{J}}$ and the space of generalized eigenvectors is spanned by the vectors $\tilde{f}(z), \tilde{f}'(z), \dots, \tilde{f}^{(\nu_a(z)-1)}(z)$.

Also formulas (65) and (66) remain valid in the same form under the assumptions of Proposition 30.

The following example has been treated in [16, Sec. 5] in the real case. Let us revisit the example allowing the parameters to be complex and using the characteristic function approach presented here. We shall follow the standard notation for basic hypergeometric series as in [7].

Example 31. Let $\lambda_n := q^n$ and $w_n := \beta q^{n/2}$, for $n \in \mathbb{Z}$, where $q, \beta \in \mathbb{C}$, $0 < |q| < 1$, and $\beta \neq 0$. Then we can put $\gamma_n = \beta^{1/2} q^{(2n-1)/8}$ for all $n \in \mathbb{Z}$.

It has been proved in [16, Prop. 15] that

$$\mathfrak{F} \left(\left\{ \frac{w}{q^{-(\nu+k)/2} - q^{(\nu+k)/2}} \right\}_{k=0}^{\infty} \right) = {}_0\phi_1(; q^{\nu}; q, -q^{\nu+1/2} w^2), \quad (85)$$

providing $0 < q < 1$, $w, \nu \in \mathbb{C}$, and $q^{\nu} \notin q^{-\mathbb{N}_0}$. However, the corresponding derivation works with no change even if we assume that $q \in \mathbb{C}$, $0 < |q| < 1$. From (85), one deduces

$$\mathfrak{F} \left(\left\{ \frac{\gamma_k^2}{z - \lambda_k} \right\}_{k=n}^{\infty} \right) = \mathfrak{F} \left(\left\{ \frac{\beta q^{(2k-1)/4}}{z - q^k} \right\}_{k=n}^{\infty} \right) = {}_0\phi_1(-; z^{-1} q^n; q, -q^n z^{-2} \beta^2). \quad (86)$$

By sending $n \rightarrow \infty$ in (86), one proves in the same way as found in the proof of [16, Lem. 18] that

$$F_{\mathcal{J}}(z) = (-\beta^2 z^{-1}; q)_{\infty}, \quad \forall z \notin q^{\mathbb{Z}} \cup \{0\}.$$

The assumptions of Proposition 30 are fulfilled with $p = 1$ and the regularizing functions are

$$\Phi_1^+(z) = (qz^{-1}; q)_{\infty} \quad \text{and} \quad \Psi_1^-(z) = (z; q)_{\infty}.$$

Altogether, one gets

$$\tilde{F}_{\mathcal{J}}(z) = (z, qz^{-1}, -\beta^2 z^{-1}; q)_{\infty}, \quad \forall z \in \mathbb{C} \setminus \{0\},$$

and, in virtue of (86), one obtains

$$\tilde{f}_n(z) = z^{-n} \beta^n q^{n(n-1)/4} (z^{-1} q^{n+1}; q)_{\infty} {}_0\phi_1(-; z^{-1} q^{n+1}; q, -q^{n+1} z^{-2} \beta^2), \quad (87)$$

for all $z \in \mathbb{C} \setminus \{0\}$.

According to Proposition 30, one has

$$\text{spec}(J) = \{0\} \cup q^{\mathbb{Z}} \cup (-\beta^2) q^{\mathbb{N}_0},$$

where all the nonzero spectral points are eigenvalues with the corresponding eigenvector being determined by (87) where z is replaced by the respective eigenvalue. It can be even shown that 0 is not an eigenvalue of J , see the proof of [16, Prop. 24]. If $-\beta^2 \notin q^{\mathbb{Z}}$, then all the eigenvalues of J have the algebraic multiplicity equal to 1, while if $-\beta^2 \in q^{\mathbb{Z}}$, then there are infinitely many eigenvalues of J with the algebraic multiplicity equal to 2.

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